SPECIAL LAGRANGIAN CONIFOLDS, I: MODULI SPACES (EXTENDED VERSION)

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Abstract. This is the extended version of the paper [20], which discusses the deformation theory of special Lagrangian (SL) conifolds in $\mathbb{C}^m$. Conifolds are a key ingredient in the compactification problem for moduli spaces of compact SLs in Calabi-Yau manifolds. The conifold category allows for the simultaneous presence of conical singularities and of non-compact, asymptotically conical, ends.

Our main theorem is the natural next step in the chain of results initiated by McLean [16] and continued by the author [18] and Joyce [11]. We survey all these results, providing a unified framework for studying the various cases and emphasizing analogies and differences. Compared to [20], this paper contains more detail but the same results. The paper also lays down the geometric foundations for our paper [21] concerning gluing constructions for SL conifolds in $\mathbb{C}^m$.

Contents

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1. Introduction

Let M be a Calabi-Yau (CY) manifold. Roughly speaking, a submanifold $L \subset M$ is *special Lagrangian* (SL) if it is both minimal and Lagrangian with respect to the ambient Riemannian and symplectic structures.

From the point of view of Riemannian Geometry it is of course natural to focus on the minimality condition. It turns out that SLs are automatically volume-minimizing in their homology class. In fact, this was Harvey and Lawson's main motivation for defining and studying SLs within the general context of Calibrated Geometry [3]. This is still the most common point of view on SLs and leads to emphasizing the role of analytic and Geometric Measure Theory techniques. It also provides a connection with various classical problems in Analysis such as the Plateau problem and the study of area-minimizing cones. In many ways it is the point of view adopted here.

From the point of view of Symplectic Geometry it is instead natural to focus on the Lagrangian condition. Specifically, SLs are examples of Maslov-zero Lagrangian submanifolds. This leads to emphasizing the role of Symplectic Topology techniques, both classical (such as the h-principle and moment maps) and contemporary (such as Floer homology). An early instance of this point of view is the work of Audin [1]; it also permeates the paper [6] by Haskins and the author.

Given this richness of ingredients it is perhaps not surprising that SLs are conjectured to play an important role in Mirror Symmetry [14], [23] and to produce interesting new invariants of CY manifolds [7]. Likewise, and more intrinsically, they also tend to exhibit other nice technical features. In particular it is by now well understood that SLs often generate smooth, finite-dimensional, moduli spaces. This SL deformation problem has been studied by a number of authors under various topological and geometric assumptions. One clear path is the chain of results initiated by McLean [16], who studied deformations of smooth compact SLs; continued by the author [18] and Marshall [15], who adapted that set-up to study certain smooth non-compact (*asymptotically conical*, AC) SLs; and further advanced by Joyce, who presented analogous results for compact *conically singular* (CS) SLs [11].

The above three classes of SLs are intimately linked, as follows. One of the main open questions in SL geometry is how to compactify McLean's moduli spaces. This problem is currently one of the biggest obstructions to progress on the above conjectures. Roughly speaking, compactifying the moduli space requires adding to it a "boundary" containing singular compact SLs. By definition, CS SLs have isolated singularities modelled on SL cones in $\mathbb{C}^m$: they would be the simplest objects appearing in this boundary. If a CS SL appears in the boundary, it must be a limit of a 1-parameter family of smooth compact SLs. These smooth SLs can be recovered via a gluing construction which desingularizes the CS SL: (i) each singularity of the CS SL defines a SL cone in $\mathbb{C}^m$; (ii) each of these cones must admit a 1-parameter family of SL desingularizations, *i.e.* AC SLs in $\mathbb{C}^m$ converging to the cone as the parameter t tends to 0; (iii) the family of smooth SLs is obtained by gluing the AC SLs into a neighbourhood of the singularities of the CS SL. This picture is made precise by Joyce's gluing results [12], [13], [9]. Section 8 of [9] then shows that, in some cases and near the boundary, the compactified moduli space can be locally written as a product of moduli spaces of AC and CS SLs.

The above classes of submanifolds are special cases within the broader category of *Riemannian conifolds*, which includes manifolds exhibiting both AC and CS ends. In other words, it allows CS SLs to become non-compact by allowing the presence of AC ends. This is of fundamental importance for the construction of SLs in $\mathbb{C}^m$: it is well-known that $\mathbb{C}^m$ does not admit any compact (smooth or singular) volume-minimizing submanifolds. Cones in $\mathbb{C}^m$ with an isolated singularity at the origin are the simplest example of conifold: the construction of new examples and the study of their properties is currently one of the most active areas of

SL research [3], [4], [5], [6], [8], [17]. Conifolds provide the appropriate framework in which to extend all the above research. In particular, they might also substitute AC SLs in Joyce's gluing results: one could try to cut out a conical singularity of the CS SL and replace it with a different singular conifold, thus jumping from one area of the boundary of the compactified moduli space, containing certain CS SLs, to another.

The paper at hand is Part I of a multi-step project aiming to set up a general theory of SL conifolds. Two other papers related to this project are currently available: [19], [21] (see also [20]). Further work is in progress. The goal of this paper is to provide a general deformation theory of SL conifolds in $\mathbb{C}^m$. The best set-up for the SL deformation problem is the one provided by Joyce [11]. It is based on his Lagrangian neighbourhood and regularity theorems [10]. Joyce's framework has two benefits: (i) it simplifies the Analysis via a reduction from the semi-elliptic operator $d \oplus d^*$ on 1-forms to the elliptic Laplace operator on functions, (ii) it nicely emphasizes the separate contributions to the dimension of $\mathcal{M}_L$ coming from the topological and from the analytic components. Along with the main result Theorem 8.8 concerning moduli spaces of CS/AC SL submanifolds in $\mathbb{C}^m$, we thus present new proofs of the previously-known results, emphasizing this point of view. In this sense, this paper also serves the purpose of surveying and unifying those results. More importantly, it lays down the geometric foundations for [21]; the analytic foundations are provided by [19].

We now summarize the contents of this paper. Section 2 introduces the category of m-dimensional Riemannian conifolds. The main definitions are standard but Section 2.2 contains an investigation into the structure of various spaces of closed 1-forms on these manifolds. This is a fundamental component of the Lagrangian and SL deformation theory. The corresponding notion of "subconifolds" is presented in Section 3, leading to the concept of *Lagrangian conifolds*. Deformation theory begins in Section 4. From various points of view it seems most satisfying to begin with the general (infinite-dimensional) theory of Lagrangian deformations. This is presented as a direct consequence of Joyce's Lagrangian neighbourhood theorems, coupled with the material of Section 2.2. The case of Lagrangian cones is studied in particular detail in Section 4.2 as it provides the backbone for all other cases. After presenting the necessary definitions in Section 5, the analogous framework for deforming SL conifolds is developed in Section 6. With the aim of making this paper reasonably self-contained, Section 7 summarizes from [19] some results concerning harmonic functions on conifolds. The SL deformation theory is then completed in Section 8. The proofs rely upon a fair amount of analytic machinery: weighted Sobolev spaces, embedding theorems and the theory of elliptic operators on conifolds. Full details are provided in [19].

Important remark: To simplify certain arguments, throughout this paper we assume $m \geq 3$.

2. Geometry of conifolds

2.1. **Asymptotically conical and conically singular manifolds.** We introduce here the categories of differentiable and Riemannian manifolds mainly relevant to this paper, referring to [19] for further details. Following [10], however, we introduce a small variation of the notion of "conically singular" manifolds: presenting them in terms of the compactification $\bar{L}$ will allow us to keep track of the singular points x_i. This plays no role in this section but in Section 4 it will become very useful.

Definition 2.1. Let L^m be a smooth manifold. We say L is a *manifold with ends* if it satisfies the following conditions:

(1) We are given a compact subset $K \subset L$ such that $S := L \setminus K$ has a finite number of connected components $S_1, \ldots, S_e$, *i.e.* $S = \amalg_{i=1}^e S_i$.

(2) For each S_i we are given a connected $(m-1)$-dimensional compact manifold Σ_i without boundary.
(3) There exist diffeomorphisms $\phi_i : \Sigma_i \times [1, \infty) \to \overline{S_i}$.

We then call the components S_i the *ends* of L and the manifolds Σ_i the *links* of L. We denote by S the union of the ends and by Σ the union of the links of L.

Definition 2.2. Let L be a manifold with ends. Let g be a Riemannian metric on L. Choose an end S_i with corresponding link Σ_i.

We say that S_i is a *conically singular* (CS) end if the following conditions hold:

(1) Σ_i is endowed with a Riemannian metric g_i'.
   We then let (θ, r) denote the generic point on the product manifold $C_i := \Sigma_i \times (0, \infty)$ and $\tilde{g}_i := dr^2 + r^2 g_i'$ denote the corresponding *conical metric* on C_i.
(2) There exist a constant $\nu_i > 0$ and a diffeomorphism $\phi_i : \Sigma_i \times (0, \epsilon] \to \overline{S_i}$ such that, as $r \to 0$ and for all $k \geq 0$,

$$|\widetilde{\nabla}^k (\phi_i^* g - \tilde{g}_i)|_{\tilde{g}_i} = O(r^{\nu_i - k}),$$

   where $\widetilde{\nabla}$ is the Levi-Civita connection on C_i defined by $\tilde{g}_i$.

We say that S_i is an *asymptotically conical* (AC) end if the following conditions hold:

(1) Σ_i is endowed with a Riemannian metric g_i'.
   We again let (θ, r) denote the generic point on the product manifold $C_i := \Sigma_i \times (0, \infty)$ and $\tilde{g}_i := dr^2 + r^2 g_i'$ denote the corresponding conical metric on C_i.
(2) There exist a constant $\nu_i < 0$ and a diffeomorphism $\phi_i : \Sigma_i \times [R, \infty) \to \overline{S_i}$ such that, as $r \to \infty$ and for all $k \geq 0$,

$$|\widetilde{\nabla}^k (\phi_i^* g - \tilde{g}_i)|_{\tilde{g}_i} = O(r^{\nu_i - k}),$$

   where $\widetilde{\nabla}$ is the Levi-Civita connection on C_i defined by $\tilde{g}_i$.

In either of the above situations we call ν_i the *convergence rate* of S_i.

We refer to [19] Section 6 for a better understanding of the asymptotic conditions introduced in Definition 2.2.

Definition 2.3. Let $(\bar{L}, d)$ be a metric space. $\bar{L}$ is a *Riemannian manifold with conical singularities* (CS manifold) if it satisfies the following conditions.

(1) We are given a finite number of points $\{x_1, \ldots, x_e\} \in \bar{L}$ such that $L := \bar{L} \setminus \{x_1, \ldots, x_e\}$ has the structure of a smooth m-dimensional manifold with e ends.
   More specifically, we assume given $\epsilon \in (0, 1)$ such that any pair of distinct points satisfies $d(x_i, x_j) > 2\epsilon$. Set $S_i := \{x \in L : 0 < d(x, x_i) < \epsilon\}$. We then assume that S_i are the ends of L with respect to some given connected links Σ_i.
(2) We are given a Riemannian metric g on L inducing the distance d.
(3) With respect to g, each end S_i is CS in the sense of Definition 2.2.

It follows from our definition that any CS manifold $\bar{L}$ is compact. We will often not distinguish between $\bar{L}$ and L, but notice that (L, g) is neither compact nor complete. We call x_i the *singularities* of $\bar{L}$.

Definition 2.4. Let (L, g) be a Riemannian manifold. L is a *Riemannian manifold with asymptotically conical ends* (AC manifold) if it satisfies the following conditions.

(1) L is a smooth manifold with e ends S_i and connected links Σ_i.
(2) Each end S_i is AC in the sense of Definition 2.2.

One can check that AC manifolds are non-compact but complete.

Definition 2.5. Let $(\bar{L}, d)$ be a metric space. We say that $\bar{L}$ is a *Riemannian CS/AC manifold* if it satisfies the following conditions.

(1) We are given a finite number of points $\{x_1, \dots, x_s\}$ and a number l such that $L := \bar{L} \setminus \{x_1, \dots, x_s\}$ has the structure of a smooth m-dimensional manifold with $s + l$ ends.
(2) We are given a metric g on L inducing the distance d.
(3) With respect to g, neighbourhoods of the points x_i have the structure of CS ends in the sense of Definition 2.2. These are the "small" ends. We also assume that the remaining ends are "large", *i.e.* they have the structure of AC ends in the sense of Definition 2.2.

We will denote the union of the CS links (respectively, of the CS ends) by Σ_0 (respectively, S_0) and those corresponding to the AC links and ends by Σ_∞, S_∞.

Definition 2.6. We use the generic term *conifold* to indicate any CS, AC or CS/AC manifold. If (L, g) is a conifold and $C := \amalg C_i$ is the union of the corresponding cones as in Definition 2.2, endowed with the induced metric $\tilde{g}$, we say that (L, g) is *asymptotic* to $(C, \tilde{g})$.

Remark 2.7. If we think of $\bar{L}$ as a generic compactification of the manifold with ends L, we should allow several CS ends to become connected by the addition of a single singular point. Notice however that we have imposed that our links be connected. We should thus allow that our points x_i be not necessarily distinct. This apparent detail becomes extremely relevant when working with "parametric connect sums", as in [19], [21]. In [19], however, we do not need to mention it because there the connect sum L_t is defined in terms of L: in some sense, the compactification $\bar{L}$ appears only *a posteriori* with respect to the connect sum, as the limit of L_t as $t \to 0$. In [21] we again do not need to mention it, this time because the connect sum is defined in terms of an immersion: by definition, the immersion is allowed to identify points so we might as well assume that the x_i and cones are initially distinct. The connect sum then depends only on the identifications determined by the immersion.

Cones in $\mathbb{R}^n$ are of course the archetype of CS/AC manifold, as follows.

Definition 2.8. A subset $\bar{\mathcal{C}} \subseteq \mathbb{R}^n$ is a *cone* if it is invariant under dilations of $\mathbb{R}^n$, *i.e.* if $t \cdot \bar{\mathcal{C}} \subseteq \bar{\mathcal{C}}$, for all $t \geq 0$. It is uniquely identified by its *link* $\Sigma := \bar{\mathcal{C}} \bigcap \mathbb{S}^{n-1}$. We will set $\mathcal{C} := \bar{\mathcal{C}} \setminus 0$. The cone is *regular* if Σ is smooth. From now on we will always assume this.

Let g' denote the induced metric on Σ. Then $\mathcal{C}$ with its induced metric is isometric to $\Sigma \times (0, \infty)$ with the conical metric $\tilde{g} := dr^2 + r^2 g'$. In particular $\bar{\mathcal{C}}$ is a CS/AC manifold; it has as many AC and CS ends as the number of connected components Σ_i of Σ. Each Σ_i thus defines a singular point x_i but these singular points are not distinct: they all coincide with the origin. Notice that Σ is a subsphere $\mathbb{S}^{m-1} \subseteq \mathbb{S}^{n-1}$ iff $\bar{\mathcal{C}}$ is an m-plane in $\mathbb{R}^n$.

Let E be a vector bundle over (L, g). Assume E is endowed with a metric and metric connection ∇: we say that (E, ∇) is a *metric pair*. In later sections E will usually be a bundle of differential forms Λ^r on L, endowed with the metric and Levi-Civita connection induced from g. We can define two types of Banach spaces of sections of E, referring to [19] for further details regarding the structure and properties of these spaces.

Regarding notation, given a vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_e) \in \mathbb{R}^e$ and $j \in \mathbb{N}$ we set $\boldsymbol{\beta} + j := (\beta_1 + j, \dots, \beta_e + j)$. We write $\boldsymbol{\beta} \geq \boldsymbol{\beta}'$ iff $\beta_i \geq \beta_i'$.

Definition 2.9. Let (L, g) be a conifold with e ends. We say that a smooth function $\rho : L \to (0, \infty)$ is a *radius function* if $\rho(x) \equiv r$ on each end. Given any vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_e) \in \mathbb{R}^e$, choose a function $\boldsymbol{\beta}$ on L which, on each end S_i, restricts to the constant β_i.

Given any metric pair (E, ∇), the *weighted Sobolev spaces* are defined by

$$W^p_{k;\boldsymbol{\beta}}(E) := \text{Banach space completion of the space } \{\sigma \in C^\infty(E) : \|\sigma\|_{W^p_{k;\boldsymbol{\beta}}} < \infty\}, \tag{2.1}$$

where we use the norm $\|\sigma\|_{W^p_{k;\boldsymbol{\beta}}} := (\Sigma_{j=0}^k \int_L |\rho^{-\boldsymbol{\beta}+j}\nabla^j\sigma|^p \rho^{-m}\operatorname{vol}_g)^{1/p}$.

The *weighted spaces of C^k sections* are defined by

$$C^k_{\boldsymbol{\beta}}(E) := \{\sigma \in C^k(E) : \|\sigma\|_{C^k_{\boldsymbol{\beta}}} < \infty\}, \tag{2.2}$$

where we use the norm $\|\sigma\|_{C^k_{\boldsymbol{\beta}}} := \sum_{j=0}^k \sup_{x\in L}|\rho^{-\boldsymbol{\beta}+j}\nabla^j\sigma|$. Equivalently, $C^k_{\boldsymbol{\beta}}(E)$ is the space of sections $\sigma \in C^k(E)$ such that $|\nabla^j\sigma| = O(r^{\boldsymbol{\beta}-j})$ as $r \to 0$ (respectively, $r \to \infty$) along each CS (respectively, AC) end. These are also Banach spaces.

To conclude, the *weighted space of smooth sections* is defined by

$$C^\infty_{\boldsymbol{\beta}}(E) := \bigcap_{k\geq 0} C^k_{\boldsymbol{\beta}}(E).$$

Equivalently, this is the space of smooth sections such that $|\nabla^j\sigma| = O(\rho^{\boldsymbol{\beta}-j})$ for all $j \geq 0$. This space has a natural Fréchet structure.

When E is the trivial $\mathbb{R}$ bundle over L we obtain weighted spaces of functions on L. We usually denote these by $W^p_{k,\boldsymbol{\beta}}(L)$ and $C^k_{\boldsymbol{\beta}}(L)$. In the case of a CS/AC manifold we will often separate the CS and AC weights, writing $\boldsymbol{\beta} = (\boldsymbol{\mu}, \boldsymbol{\lambda})$ for some $\boldsymbol{\mu} \in \mathbb{R}^s$ and some $\boldsymbol{\lambda} \in \mathbb{R}^l$. We then write $C^k_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(E)$ and $W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(E)$.

For these spaces one can prove the validity of the following weighted version of the Sobolev Embedding Theorems, cf. [19].

Theorem 2.10. *Let (L, g) be an AC manifold. Let (E, ∇) be a metric pair over L. Assume $k \geq 0$, $l \in \{1, 2, \dots\}$ and $p \geq 1$. Set $p^*_l := \frac{mp}{m-lp}$. Then, for all $\boldsymbol{\beta}' \geq \boldsymbol{\beta}$,*

1. *If $lp < m$ then there exists a continuous embedding $W^p_{k+l,\boldsymbol{\beta}}(E) \hookrightarrow W^{p^*_l}_{k,\boldsymbol{\beta}'}(E)$.*
2. *If $lp = m$ then, for all $q \in [p, \infty)$, there exist continuous embeddings $W^p_{k+l,\boldsymbol{\beta}}(E) \hookrightarrow W^q_{k,\boldsymbol{\beta}'}(E)$.*
3. *If $lp > m$ then there exists a continuous embedding $W^p_{k+l,\boldsymbol{\beta}}(E) \hookrightarrow C^k_{\boldsymbol{\beta}'}(E)$.*

Furthermore, assume $kp > m$. Then the corresponding weighted Sobolev spaces are closed under multiplication, in the following sense. For any $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ there exists $C > 0$ such that, for all $u \in W^p_{k,\boldsymbol{\beta_1}}$ and $v \in W^p_{k,\boldsymbol{\beta_2}}$,

$$\|uv\|_{W^p_{k,\boldsymbol{\beta_1}+\boldsymbol{\beta_2}}} \leq C\|u\|_{W^p_{k,\boldsymbol{\beta_1}}}\|v\|_{W^p_{k,\boldsymbol{\beta_2}}}.$$

Let (L, g) be a CS manifold. Then the same conclusions hold for all $\boldsymbol{\beta}' \leq \boldsymbol{\beta}$.

Let (L, g) be a CS/AC manifold. Then, setting $\boldsymbol{\beta} = (\boldsymbol{\mu}, \boldsymbol{\lambda})$, the same conclusions hold for $\boldsymbol{\mu}' \leq \boldsymbol{\mu}$ on the CS ends and $\boldsymbol{\lambda}' \geq \boldsymbol{\lambda}$ on the AC ends.

2.2. Cohomology of manifolds with ends.

Any smooth compact manifold or smooth manifold with ends L has topology of finite type. In particular, the first cohomology group

$$H^1(L; \mathbb{R}) := \frac{\{\text{Smooth closed 1-forms on } L\}}{d(C^\infty(L))}$$

has finite dimension $b^1(L)$, proving the following statement concerning the structure of the space of smooth closed 1-forms.

Decomposition 1 (for compact manifolds or manifolds with ends)**.** Let L be a smooth compact manifold or a smooth manifold with ends. Choose a finite-dimensional vector space H of closed 1-forms on L such that the map

$$H \to H^1(L; \mathbb{R}), \quad \alpha \mapsto [\alpha] \tag{2.3}$$

is an isomorphism. Then

$$\{\text{Smooth closed 1-forms on } L\} = H \oplus d(C^\infty(L)). \tag{2.4}$$

We now want to show that in the case of a manifold with ends there exist natural conditions on the space of 1-forms H.

Definition 2.11. Given a manifold Σ, set $C := \Sigma \times (0,\infty)$. Consider the projection $\pi : \Sigma \times (0,\infty) \to \Sigma$. A p-form η on C is *translation-invariant* if it is of the form $\eta = \pi^*\eta'$, for some p-form η' on Σ.

Lemma 2.12. *Let L be a smooth manifold with ends S_i. Let α be a smooth closed 1-form on L. Then there exist a smooth closed 1-form α' and a smooth function A on L such that $\alpha'_{|S_i}$ is translation-invariant and $\alpha = \alpha' + dA$. If furthermore α has compact support then we can choose α' to have compact support.*

Proof. The proof follows the scheme of the Poincaré Lemma for de Rham cohomology, cf. *e.g.* [2]. Given any p-form η on $S_i = \Sigma_i \times (1,\infty)$, we can write

$$\eta = \eta_1(\theta, r) + \eta_2(\theta, r) \wedge dr$$

for some r-dependent p-form η_1 and $(p-1)$-form η_2 on Σ. Specifically, η_1 is the restriction of η to the cross-sections $\Sigma_i \times \{r\}$ and $\eta_2 := i_{\partial r}\eta$. For a fixed $R_0 > 1$ we then define $(K\eta)(\theta, r) := \int_{R_0}^r \eta_2(\theta,\rho)\, d\rho$.

Let us apply this to the 1-form obtained by restricting α to S_i, writing

$$\alpha_{|S_i} = \alpha_1(\theta, r) + \alpha_2(\theta, r)\, dr$$

for some r-dependent 1-form α_1 and function α_2 on Σ_i. It is then easy to check that

$$\begin{aligned}
d\alpha_{|S_i} &= d_\Sigma \alpha_1 - (\frac{\partial}{\partial r}\alpha_1) \wedge dr + (d_\Sigma \alpha_2) \wedge dr, \\
K\alpha_{|S_i} &= \int_{R_0}^r \alpha_2(\theta,\rho)\, d\rho, \\
d(K\alpha_{|S_i}) &= \int_{R_0}^r d_\Sigma \alpha_2(\theta,\rho)\, d\rho + \alpha_2(\theta, r)\, dr.
\end{aligned}$$

From $d\alpha = 0$ it follows that $\alpha_1(\theta, R_0) + d(K\alpha) = \alpha_{|S_i}$ and that $\alpha_1(\theta, R_0)$ is closed. Setting $\alpha'_i := \alpha_1(\theta, R_0)$ and $A_i := K\alpha$ we can rewrite this as $\alpha_{|S_i} = \alpha'_i + dA_i$. Interpolating between the A_i yields a global smooth function A on L such that $\alpha_{|S_i} = \alpha'_i + dA_{|S_i}$. We can now define $\alpha' := \alpha - dA$ to obtain the global relationship

$$\alpha = \alpha' + dA.$$

It is clear from this construction that if α has compact support then (choosing R_0 large enough) α' also has compact support. $\square$

Recall that compactly-supported forms give rise to the following theory. Let L be a smooth manifold with ends. We denote by $\Lambda^p_c(L;\mathbb{R})$ the space of smooth compactly-supported p-forms on L and by $H^p_c(L;\mathbb{R})$ the corresponding cohomology groups. Let Σ denote the union of the links of L. Notice that L is deformation-equivalent to a compact manifold with boundary Σ. Standard algebraic topology (see also [10] Section 2.4) proves that the inclusion $\Sigma \subset L$ gives rise to a long exact sequence in cohomology

$$0 \to H^0(L;\mathbb{R}) \to H^0(\Sigma;\mathbb{R}) \xrightarrow{\delta} H^1_c(L;\mathbb{R}) \xrightarrow{\gamma} H^1(L;\mathbb{R}) \xrightarrow{\rho} H^1(\Sigma;\mathbb{R}) \to \dots. \tag{2.5}$$

Here, γ is induced by the injection $\Lambda^1_c(L;\mathbb{R}) \to \Lambda^1(L;\mathbb{R})$ and ρ is induced by the restriction $\Lambda^1(L;\mathbb{R}) \to \Lambda^1(\Sigma;\mathbb{R})$. We set $\widetilde{H}^1_c := \mathrm{Im}(\gamma) = \mathrm{Ker}(\rho)$. Exactness implies that

$$\begin{aligned}\dim(\widetilde{H}^1_c) &= \dim(H^1_c(L;\mathbb{R})) - \dim(H^0(\Sigma;\mathbb{R})) + \dim(H^0(L;\mathbb{R})) \\ &= b^1_c(L) - e + 1.\end{aligned} \tag{2.6}$$

Remark 2.13. The sequence 2.5 shows that

$$H^1_c(L,\mathbb{R}) \simeq \widetilde{H}^1_c \oplus \mathrm{Ker}(\gamma) = \widetilde{H}^1_c \oplus \mathrm{Im}(\delta). \tag{2.7}$$

This decomposition can be expressed in words as follows. By definition, $H^1_c(L;\mathbb{R})$ is determined by the classes of compactly-supported 1-forms which are not the differential of a compactly-supported function. Given any such form, there are two cases: (i) it is not the differential of *any* function, in which case γ maps its class to a non-zero element of $\widetilde{H}^1_c$, (ii) it is the differential of some function, in which case γ maps its class to zero. However, this function is necessarily constant on the ends of L: these constants can be parametrized via $H^0(\Sigma;\mathbb{R})$. Notice that the function is only well-defined up to a constant; likewise, $\mathrm{Im}(\delta)$ coincides with $H^0(\Sigma;\mathbb{R})$ only up to $H^0(L;\mathbb{R}) \simeq \mathbb{R}$.

Concerning Decomposition 1, we can now choose H as follows. For $i = 1, \dots, k = \dim(\tilde{H}^1_c)$ let $[\alpha_i]$ be a basis of $\widetilde{H}^1_c$. According to Lemma 2.12 we can choose α'_i with compact support such that $[\alpha'_i] = [\alpha_i]$. For $i = 1, \dots, N = \dim(H^1)$ let $[\alpha_i]$ denote an extension to a basis of $H^1(L;\mathbb{R})$. Again using Lemma 2.12 we can choose an extension α'_i of translation-invariant 1-forms such that $[\alpha'_i] = [\alpha_i]$. Set

$$\widetilde{H} := \mathrm{span}\{\alpha'_1, \dots, \alpha'_k\}, \quad H := \mathrm{span}\{\alpha'_1, \dots, \alpha'_N\}. \tag{2.8}$$

Then H satisfies the assumptions of Decomposition 1. One advantage of this choice of H is that it reflects the relationship of $\widetilde{H}^1_c$ to H^1. Specifically, if we apply Decomposition 1 to α writing $\alpha = \alpha' + dA$ with $\alpha' \in H$, then $[\alpha] \in \widetilde{H}^1_c$ iff $\alpha' \in \widetilde{H}$, *i.e.* iff α' has compact support.

2.3. **Cohomology of conifolds.** We now want to achieve analogous decompositions for CS and AC manifolds, in terms of weighted spaces of closed and exact 1-forms.

Lemma 2.14. *Let (Σ, g') be a Riemannian manifold. Let the corresponding cone C have the conical metric $\tilde{g} := dr^2 + r^2 g'$. Then any translation-invariant p-form $\eta = \pi^*\eta'$ belongs to the weighted space $C^\infty_{(-p,-p)}(\Lambda^p)$. For any $\beta > 0$, η belongs to the smaller weighted space $C^\infty_{(-p+\beta,-p-\beta)}(\Lambda^p)$ iff $\eta' = 0$.*

Proof. As seen in the proof of Lemma 2.12, the general p-form η on C can be written $\eta = \eta_1(\theta, r) + \eta_2(\theta, r) \wedge dr$. The form is translation-invariant iff η_1 is r-independent and $\eta_2 = 0$. In this case $|\eta|_{\tilde{g}} = r^{-p}|\eta_1|_{g'}$ so $|\eta|_{\tilde{g}} = O(r^{-p})$ both for $r \to 0$ and for $r \to \infty$. This proves that $\eta \in C^0_{(-p,-p)}(\Lambda^p)$. To show that $\eta \in C^\infty_{(-p,-p)}(\Lambda^p)$ it is necessary to estimate $|\widetilde{\nabla}^k \eta|_{\tilde{g}}$, where $\widetilde{\nabla}$ is the Levi-Civita connection. This can be done fairly explicitly in terms of Christoffel symbols. In particular one can choose local coordinates on $U \subset \Sigma$ defining a local frame $\partial_1, \cdots, \partial_{m-1}$. Set $\partial_0 := \partial r$, the standard frame on $(0, \infty)$. The Christoffel symbols for the corresponding frame on $(0, \infty) \times U$ and the metric $\tilde{g}$ can then be computed explicitly: for $i, j, k \geq 1$ one finds that $\widetilde{\Gamma}^k_{i,j}$ is bounded, $\widetilde{\Gamma}^0_{i,j} = O(r)$, $\widetilde{\Gamma}^k_{i,0} = O(r^{-1})$, $\widetilde{\Gamma}^k_{0,0} = \widetilde{\Gamma}^0_{i,0} = \widetilde{\Gamma}^0_{0,0} = 0$. The Christoffel symbols defined by $\tilde{g}$ for the other tensor bundles depend linearly on these, so they have the same bounds. Using these calculations one finds that $|\widetilde{\nabla}^k \eta|_{\tilde{g}} = O(r^{-p-k})$, as desired.

It is clear from the proof that η satisfies stronger bounds iff it vanishes. □

Decomposition 2 (for CS or AC manifolds and forms with allowable growth)**.** Let L be a CS manifold. Choose a finite-dimensional vector space H of smooth closed 1-forms on L as in Equation 2.8. Then, for any $\boldsymbol{\beta} < 0$,

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} = H \oplus d(C^\infty_{\boldsymbol{\beta}}(L)). \tag{2.9}$$

Analogously, let L be an AC manifold. Choose H as above. Then, for any $\boldsymbol{\beta} > 0$,

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} = H \oplus d(C^\infty_{\boldsymbol{\beta}}(L)). \tag{2.10}$$

Proof. Consider the CS case. Since $\boldsymbol{\beta} < 0$, Lemma 2.14 proves that $H \oplus d(C^\infty_{\boldsymbol{\beta}}(L)) \subseteq \{\text{Closed 1-forms in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\}$. Now choose a closed $\alpha \in C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$. By Decomposition 1 we can write $\alpha = \alpha' + dA$, for some $\alpha' \in H$ and $A \in C^\infty(L)$. Notice that $dA = \alpha - \alpha' \in C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$. By integration, again using the fact $\boldsymbol{\beta} < 0$, we conclude that $A \in C^\infty_{\boldsymbol{\beta}}(L)$. This proves the opposite inclusion, thus the identity. The AC case is analogous. □

Lemma 2.15. *Assume L is a CS manifold. If α is a smooth closed 1-form on L belonging to the space $C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$ for some $\boldsymbol{\beta} > 0$ then there exists a smooth closed 1-form α' with compact support on L and a smooth function $A \in C^\infty_{\boldsymbol{\beta}}(L)$ such that $\alpha = \alpha' + dA$.*

Assume L is an AC manifold. If α is a smooth closed 1-form on L belonging to the space $C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$ for some $\boldsymbol{\beta} < 0$ then there exists a smooth closed 1-form α' with compact support on L and a smooth function $A \in C^\infty_{\boldsymbol{\beta}}(L)$ such that $\alpha = \alpha' + dA$.

Proof. The proof is a variation of the proof of Lemma 2.12, as follows. Consider the AC case. Write $\alpha_{|S_i} = \alpha_1 + \alpha_2 \wedge dr$. Define $K\alpha := -\int_r^\infty \alpha_2(\theta,\rho)\,d\rho$: this converges because $\boldsymbol{\beta} < 0$. It is simple to check that $d(K\alpha) = \alpha$; in particular, this shows that α is exact on each end S_i. Setting $A := K\alpha$ and extending as in Lemma 2.12 leads to a global decomposition $\alpha = \alpha' + dA$ on L. By construction α' has compact support and $A \in C^\infty_{\boldsymbol{\beta}}$. The CS case is analogous, with $K\alpha := \int_0^r \alpha_2(\theta,\rho)\,d\rho$. □

Decomposition 3 (for CS or AC manifolds and forms with allowable decay)**.** Let L be a CS manifold. Assume $\boldsymbol{\beta} > 0$. Choose a finite-dimensional vector space H of closed 1-forms on L as in Equation 2.8, using $\widetilde{H}_0$ to denote the space $\widetilde{H}$. For any $i = 1, \dots, e$ choose a smooth function f_i on L such that $f_i \equiv 1$ on the end S_i and $f_i \equiv 0$ on the other ends. We can do this in such a way that $\sum f_i \equiv 1$. Let E_0 denote the e-dimensional vector space generated by these functions. By construction E_0 contains the constant functions so $d(E_0)$ has dimension $e-1$. It is simple to check that $d(E_0) \cap d(C^\infty_{\boldsymbol{\beta}}(L)) = \{0\}$. Then

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} = \widetilde{H}_0 \oplus d(E_0) \oplus d(C^\infty_{\boldsymbol{\beta}}(L)). \tag{2.11}$$

Analogously, let L be an AC manifold. Assume $\boldsymbol{\beta} < 0$. Choose spaces as above, this time using the notation $\widetilde{H}_\infty$ and E_∞. Then

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} = \widetilde{H}_\infty \oplus d(E_\infty) \oplus d(C^\infty_{\boldsymbol{\beta}}(L)). \tag{2.12}$$

Proof. Consider the CS case. The inclusion $\supseteq$ is clear. Conversely, let $\alpha \in C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$ be closed. Decomposition 1 allows us to write $\alpha = \alpha' + dA$, for some uniquely defined $\alpha' \in H$ and some $A \in C^\infty(L)$, well-defined up to a constant. Lemma 2.15 implies that the cohomology class of α belongs to the space $\widetilde{H}^1_c$, *i.e.* that $\alpha' \in \widetilde{H}_0$ so it has compact support. This shows that $dA \in C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$. Writing $A_i := A_{|S_i}$ we find $dA_i = d_{\Sigma_i} A_i + \frac{\partial A_i}{\partial r}\,dr$, thus $\frac{\partial A_i}{\partial r} \in C^\infty_{\boldsymbol{\beta}-1}(L)$. This shows that $\int_0^r \frac{\partial A_i}{\partial r}\,d\rho \in C^\infty_{\boldsymbol{\beta}}(L)$. This determines A_i up to a constant c_i on each end. Together with Equation 2.7 this proves the claim. The AC case is analogous. □

We now turn to the case of CS/AC manifolds, concentrating on the situations of most interest to us.

Decomposition 4 (for CS/AC manifolds)**.** Let L be a CS/AC manifold with s CS ends and l AC ends. As usual we denote the union of the CS links by Σ_0 and the union of the AC links by Σ_∞. Choose a finite-dimensional vector space H of closed 1-forms on L as in Equation 2.8, using $\widetilde{H}_{0,\infty}$ to denote the space $\widetilde{H}$. For any $i = 1, \dots, s+l$ choose a function f_i such that $f_i \equiv 1$ on the end S_i and $f_i \equiv 0$ on the other ends. We can assume that $\sum f_i \equiv 1$. Let $E_{0,\infty}$ denote the $(s+l)$-dimensional vector space generated by these functions. Then, for any $\boldsymbol{\mu} > 0$ and $\boldsymbol{\lambda} < 0$,

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)\} = \widetilde{H}_{0,\infty} \oplus d(E_{0,\infty}) \oplus d(C^\infty_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)). \tag{2.13}$$

Now let $\Lambda^p_{c,\bullet}(L;\mathbb{R})$ denote the space of p-forms on L which vanish in a neighbourhood of the singularities, with no condition on the large ends. Let $H^p_{c,\bullet}(L;\mathbb{R})$ denote the corresponding cohomology groups. Let $\widetilde{H}^1_{c,\bullet}$ denote the image of the map $\gamma : H^1_{c,\bullet}(L;\mathbb{R}) \to H^1(L;\mathbb{R})$. Choose a finite-dimensional vector space $\widetilde{H}_{0,\bullet}$ of translation-invariant closed 1-forms on L with compact support in a neighbourhood of the singularities and such that the map

$$\widetilde{H}_{0,\bullet} \to \widetilde{H}^1_{c,\bullet}, \quad \alpha \mapsto [\alpha] \tag{2.14}$$

is an isomorphism. For any $i = 1, \dots, s$ choose a function f_i such that $f_i \equiv 1$ on the CS end corresponding to the singularity x_i and $f_i \equiv 0$ on the other ends. Let E_0 denote the s-dimensional vector space generated by these functions. Then, for any $\boldsymbol{\mu} > 0$ and $\boldsymbol{\lambda} > 0$,

$$\{\text{Closed 1-forms on } L \text{ in } C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)\} = \widetilde{H}_{0,\bullet} \oplus d\Big(E_0 \oplus C^\infty_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)\Big). \tag{2.15}$$

Proof. The proof is similar to the proofs of the previous decompositions. It may however be good to emphasize that, in the case $\boldsymbol{\mu} > 0$ and $\boldsymbol{\lambda} > 0$, $d(E_0) \cap d(C^\infty_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)) \neq \{0\}$ (it is one-dimensional). This explains the slightly different statement of Decomposition 2.15. □

Remark 2.16. The weight $\boldsymbol{\beta} = 0$ corresponds to an exceptional case in Lemma 2.15: integration will generally generate log terms, so we cannot conclude that $A \in C^\infty_{\boldsymbol{\beta}}$ there. One can analogously argue that $C^\infty_{-\mathbf{1}}(\Lambda^1)/d(C^\infty_{\mathbf{0}}(L))$ is not finite-dimensional.

Similar decompositions hold for k-forms: in this setting the exceptional case corresponds to $\boldsymbol{\beta} = k-1$.

Remark 2.17. Notice that the above decompositions do not cover all possibilities: for example, given a CS manifold we could decide to study the space of closed 1-forms in $C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)$ corresponding to a weight $\boldsymbol{\beta} = (\beta_1, \dots, \beta_e)$ with some β_i positive and others negative. However, it should be clear from the above discussion how to use the same ideas to cover any other case of interest. We have restricted our attention to the cases most relevant to this paper.

For future reference it is useful to emphasize the topological interpretation of some of the previous results. The reasons underlying our interest for each case will become apparent in Section 8.

Corollary 2.18. *Let L be a smooth compact manifold. Then*

$$\{\textit{Closed 1-forms on } L\} \simeq H^1(L;\mathbb{R}) \oplus d(C^\infty(L)).$$

Let (L, g) be an AC manifold. Then for $\boldsymbol{\beta} < 0$

$$\{\textit{Closed 1-forms on } L \textit{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} \simeq H^1_c(L;\mathbb{R}) \oplus d(C^\infty_{\boldsymbol{\beta}}(L)),$$

while for $\boldsymbol{\beta} > 0$

$$\{\textit{Closed 1-forms on } L \textit{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\} \simeq H^1(L;\mathbb{R}) \oplus d(C^\infty_{\boldsymbol{\beta}}(L)).$$

Let (L,g) be a CS manifold with link Σ_0. Then for $\boldsymbol{\beta}>0$

$$\{\textit{Closed } 1\textit{-forms on } L \textit{ in } C^\infty_{\boldsymbol{\beta}-1}(\Lambda^1)\}$$
$$\simeq Ker\left(H^1(L)\stackrel{\rho}{\to}H^1(\Sigma_0)\right)\oplus d(E_0)\oplus d(C^\infty_{\boldsymbol{\beta}}(L)).$$

Let (L,g) be a CS/AC manifold with link $\Sigma=\Sigma_0\amalg\Sigma_\infty$. Then for $\boldsymbol{\mu}>0$ and $\boldsymbol{\lambda}<0$

$$\{\textit{Closed } 1\textit{-forms on } L \textit{ in } C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)\}$$
$$\simeq Ker\left(H^1_{\bullet,c}(L)\stackrel{\rho}{\to}H^1(\Sigma_0)\right)\oplus d(E_0)\oplus d(C^\infty_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)),$$

while for $\boldsymbol{\mu}>0$ and $\boldsymbol{\lambda}>0$

$$\{\textit{Closed } 1\textit{-forms on } L \textit{ in } C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)\}$$
$$\simeq Ker\left(H^1(L)\stackrel{\rho}{\to}H^1(\Sigma_0)\right)\oplus d\left(E_0\oplus C^\infty_{(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)\right).$$

Proof. The compact case coincides with Equation 2.4. The AC case with $\boldsymbol{\beta}<0$ follows from Equation 2.12 and Remark 2.13. The AC case with $\boldsymbol{\beta}>0$ coincides with Equation 2.10. The CS case coincides with Equation 2.11.

Let us now focus on the CS/AC case with $\boldsymbol{\lambda}<0$. Using the notation of Decomposition 4, let E' denote a complement of $E_0\oplus\mathbb{R}$ in $E_{0,\infty}$, *i.e.* $E_{0,\infty}=E_0\oplus\mathbb{R}\oplus E'$. Notice that the long exact sequence 2.5 with $\Sigma=\Sigma_0\amalg\Sigma_\infty$ leads to an identification $H^1_c(L;\mathbb{R})\simeq\widetilde{H}^1_c(L)\oplus d(E_{0,\infty})$. One can also set up the "relative" analogue of Sequence 2.5 using the inclusion of pairs $(\Sigma_0,\emptyset)\subset(L,\Sigma_\infty)$. Using notation analogous to that of Decomposition 4 this leads to the long exact sequence

$$0\to H^0_c(L;\mathbb{R})\to H^0_{\bullet,c}(L;\mathbb{R})\to H^0(\Sigma_0;\mathbb{R})\to H^1_c(L;\mathbb{R})\stackrel{\gamma}{\to}H^1_{\bullet,c}(L;\mathbb{R})\stackrel{\rho}{\to}H^1(\Sigma_0;\mathbb{R})\to\dots$$

Since $H^0_c(L;\mathbb{R})=0$ and $H^0_{\bullet,c}(L;\mathbb{R})=0$, one obtains an identification $H^1_c(L;\mathbb{R})\simeq E_0\oplus\mathrm{Ker}\left(H^1_{\bullet,c}(L)\stackrel{\rho}{\to}H^1(\Sigma_0)\right)$. Comparing these identifications yields an identification $\widetilde{H}^1_c(L;\mathbb{R})\oplus d(E')\simeq\mathrm{Ker}\left(H^1_{\bullet,c}(L)\stackrel{\rho}{\to}H^1(\Sigma_0)\right)$. The claim follows.

Now consider the CS/AC case with $\boldsymbol{\lambda}>0$. The long exact sequence 2.5 with $\Sigma=\Sigma_0$ yields

$$0\to H^0(L;\mathbb{R})\to H^0(\Sigma_0;\mathbb{R})\to H^1_{c,\bullet}(L;\mathbb{R})\stackrel{\gamma}{\to}H^1(L;\mathbb{R})\stackrel{\rho}{\to}H^1(\Sigma_0;\mathbb{R})\to\dots\tag{2.16}$$

This proves the final claim. □

Remark 2.19. Compare Equations 2.11, 2.12 with the corresponding equations in the statement of Corollary 2.18. When working with AC manifolds we choose to group the two topological terms of Equation 2.12 into one space $H^1_c(L;\mathbb{R})$. When working with CS manifolds we prefer to keep the two topological terms of Equation 2.11 separate and to emphasize the "geometric" meaning of one of them as kernel of a certain restriction map. These choices are based on the different roles that these spaces will play in Section 8, cf. also Remark 8.9.

3. Lagrangian conifolds

A priori, a *CS/AC submanifold* might simply be defined as an immersed submanifold whose topology and induced metric is of the type defined in Section 2.1. However, for the purposes of this article it is convenient to strengthen the hypotheses by adding the requirement that the submanifold have a well-defined cone at each singularity and at each end. The precise definitions are as follows. We restrict our attention to Lagrangian submanifolds in Kähler ambient spaces, but it is clear how one might extend these definitions to other settings.

Definition 3.1. Let (M^{2m}, ω) be a symplectic manifold. An embedded or immersed submanifold $\iota : L^m \to M$ is *Lagrangian* if $\iota^*\omega \equiv 0$. The immersion allows us to view the tangent bundle TL of L as a subbundle of TM (more precisely, of ι^*TM). When M is Kähler with structures (g, J, ω) it is simple to check that L is Lagrangian iff J maps TL to the normal bundle NL of L, *i.e.* $J(TL) = NL$.

Definition 3.2. Let L^m be a smooth manifold. Assume given a Lagrangian immersion $\iota : L \to \mathbb{C}^m$, the latter endowed with its standard structures $\tilde{J}, \tilde{\omega}$. We say that L is an *asymptotically conical Lagrangian submanifold* with *rate* $\boldsymbol{\lambda}$ if it satisfies the following conditions.

1. We are given a compact subset $K \subset L$ such that $S := L \setminus K$ has a finite number of connected components $S_1, \dots, S_e$.
2. We are given Lagrangian cones $\mathcal{C}_i \subset \mathbb{C}^m$ with smooth connected links $\Sigma_i := \mathcal{C}_i \bigcap \mathbb{S}^{2m-1}$. Let $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ denote the natural immersions, parametrizing $\mathcal{C}_i$.
3. We are finally given an e-tuple of *convergence rates* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_e)$ with $\lambda_i < 2$, *centers* $p_i \in \mathbb{C}^m$ and diffeomorphisms $\phi_i : \Sigma_i \times [R, \infty) \to \overline{S_i}$ for some $R > 0$ such that, for $r \to \infty$ and all $k \geq 0$,

$$|\widetilde{\nabla}^k(\iota \circ \phi_i - (\iota_i + p_i)| = O(r^{\lambda_i - 1 - k}) \tag{3.1}$$

   with respect to the conical metric $\tilde{g}_i$ on $\mathcal{C}_i$.

Notice that the restriction $\lambda_i < 2$ ensures that the cone is unique but is weak enough to allow the submanifold to converge to a translated copy $\mathcal{C}_i + p_i'$ of the cone (*e.g.* if $\lambda_i = 1$), or even to slowly pull away from the cone (if $\lambda_i > 1$).

Definition 3.3. Let $\bar{L}^m$ be a smooth manifold except for a finite number of possibly singular points $\{x_1, \dots, x_e\}$. Assume given a continuous map $\iota : \bar{L} \to \mathbb{C}^m$ which restricts to a smooth Lagrangian immersion of $L := \bar{L} \setminus \{x_1, \dots, x_e\}$. We say that $\bar{L}$ (or L) is a *conically singular Lagrangian submanifold* with *rate* $\boldsymbol{\mu}$ if it satisfies the following conditions.

1. We are given open connected neighbourhoods S_i of x_i.
2. We are given Lagrangian cones $\mathcal{C}_i \subset \mathbb{C}^m$ with smooth connected links $\Sigma_i := \mathcal{C}_i \bigcap \mathbb{S}^{2m-1}$. Let $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ denote the natural immersions, parametrizing $\mathcal{C}_i$.
3. We are finally given an e-tuple of *convergence rates* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_e)$ with $\mu_i > 2$, *centers* $p_i \in \mathbb{C}^m$ and diffeomorphisms $\phi_i : \Sigma_i \times (0, \epsilon] \to \overline{S_i} \setminus \{x_i\}$ such that, for $r \to 0$ and all $k \geq 0$,

$$|\widetilde{\nabla}^k(\iota \circ \phi_i - (\iota_i + p_i))| = O(r^{\mu_i - 1 - k}) \tag{3.2}$$

   with respect to the conical metric $\tilde{g}_i$ on $\mathcal{C}_i$. Notice that our assumptions imply that $\iota(x_i) = p_i$.

It is simple to check that AC Lagrangian submanifolds, with the induced metric, satisfy Definition 2.4 with $\nu_i = \lambda_i - 2$. The analogous fact holds for CS Lagrangian submanifolds.

Definition 3.4. Let $\bar{L}^m$ be a smooth manifold except for a finite number of possibly singular points $\{x_1, \dots, x_s\}$ and with l ends. Assume given a continuous map $\iota : \bar{L} \to \mathbb{C}^m$ which restricts to a smooth Lagrangian immersion of $L := \bar{L} \setminus \{x_1, \dots, x_s\}$. We say that $\bar{L}$ (or L) is a *CS/AC Lagrangian submanifold* with *rate* $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ if in a neighbourhood of the points x_i it has the structure of a CS submanifold with rates μ_i and in a neighbourhood of the remaining ends it has the structure of an AC submanifold with rates λ_i.

We use the generic term *Lagrangian conifold* (even though "subconifold" would be more appropriate) to indicate any CS, AC or CS/AC Lagrangian submanifold.

Example 3.5. Let $\mathcal{C}$ be a cone in $\mathbb{C}^m$ with smooth link Σ^{m-1}. It can be shown that $\mathcal{C}$ is a Lagrangian iff Σ is *Legendrian* in $\mathbb{S}^{2m-1}$ with respect to the natural *contact structure* on the sphere. Then $\mathcal{C}$ is a CS/AC Lagrangian submanifold of $\mathbb{C}^m$ with rate $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ for any $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$.

The definition of CS Lagrangian submanifolds can be generalized to Kähler ambient spaces as follows. Once again we denote the standard structures on $\mathbb{C}^m$ by $\tilde{J}$, $\tilde{\omega}$.

Definition 3.6. Let (M^{2m}, J, ω) be a Kähler manifold and $\bar{L}^m$ be a smooth manifold except for a finite number of possibly singular points $\{x_1, \dots, x_e\}$. Assume given a continuous map $\iota : \bar{L} \to M$ which restricts to a smooth Lagrangian immersion of $L := \bar{L} \setminus \{x_1, \dots, x_e\}$. We say that $\bar{L}$ (or L) is a *Lagrangian submanifold with conical singularities* (CS Lagrangian submanifold) if it satisfies the following conditions.

(1) We are given isomorphisms $\upsilon_i : \mathbb{C}^m \to T_{\iota(x_i)}M$ such that $\upsilon_i^*\omega = \tilde{\omega}$ and $\upsilon_i^* J = \tilde{J}$.
According to Darboux' theorem, cf. *e.g.* [24], there then exist an open ball B_R in $\mathbb{C}^m$ (of small radius R) and diffeomorphisms $\Upsilon_i : B_R \to M$ such that $\Upsilon(0) = \iota(x_i)$, $d\Upsilon_i(0) = \upsilon_i$ and $\Upsilon_i^*\omega = \tilde{\omega}$.

(2) We are given open neighbourhoods S_i of x_i in $\bar{L}$. We assume S_i are small, in the sense that the compositions
$$\Upsilon_i^{-1} \circ \iota : S_i \to B_R$$
are well-defined.
We are also given Lagrangian cones $\mathcal{C}_i \subset \mathbb{C}^m$ with smooth connected links $\Sigma_i := \mathcal{C}_i \bigcap \mathbb{S}^{2m-1}$. Let $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ denote the natural immersions, parametrizing $\mathcal{C}_i$.

(3) We are finally given an e-tuple of *convergence rates* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_e)$ with $\mu_i \in (2, 3)$ and diffeomorphisms $\phi_i : \Sigma_i \times (0, \epsilon] \to \overline{S_i} \setminus \{x_i\}$ such that, as $r \to 0$ and for all $k \geq 0$,
$$|\widetilde{\nabla}^k(\Upsilon_i^{-1} \circ \iota \circ \phi_i - \iota_i)| = O(r^{\mu_i - 1 - k}) \tag{3.3}$$
with respect to the conical metric $\tilde{g}_i$ on $\mathcal{C}_i$.

We call x_i the *singularities* of $\bar{L}$ and υ_i the *identifications*.

One can check that, when $M = \mathbb{C}^m$, Definition 3.6 coincides with Definition 3.3 if we choose $\Upsilon_i(x) := x + \iota(x_i)$. Notice that the local diffeomorphisms between M and $\mathbb{C}^m$ are prescribed only up to first order. Changing the diffeomorphism Υ_i (while keeping υ_i fixed) will perturb the map ϕ_i (and its derivatives) by a term of order $O(r^{2-k})$. In order to make the rate be independent of the particular diffeomorphism chosen, we need to introduce a constraint on the range of μ_i ensuring that $O(r^{2-k}) < O(r^{\mu_i - 1 - k})$, thus $\mu_i < 3$.

Remark 3.7. One could also define and study AC Lagrangian submanifolds in M, but this would require a preliminary study of AC metrics on Kähler manifolds, going beyond the scope of this article. We refer to [18] for some details in this direction.

4. Deformations of Lagrangian conifolds

We now want to understand how to parametrize the *Lagrangian deformations* of a given Lagrangian conifold $L \subset M$. Since the Lagrangian condition is invariant under reparametrization of L, to avoid huge amounts of geometric redundancy it is best to work in terms of non-parametrized submanifolds; in other words, in terms of equivalence classes of immersed submanifolds, where two immersions are *equivalent* if they differ by a reparametrization. Then, to parametrize the possible deformations of L, it is sufficient to prove a *Lagrangian neighbourhood theorem*.

Remark 4.1. The analogous situation in the Riemannian setting is well-known. The set $\mathrm{Imm}(L, M)$ of immersions $L \to (M, g)$ can be topologized via the C^1 or *Whitney* topology, *i.e.* in terms of the natural topology on the first jet bundle $J^1(L, M)$. The group of diffeomorphisms $\mathrm{Diff}(L)$ acts on this space by reparametrization. Choose an element $\iota \in \mathrm{Imm}(L, M)$.

Let NL denote the normal bundle. Using the *tubular neighbourhood theorem* one can define a natural injection

$$\Lambda^0(NL) \to \mathrm{Imm}(L,M)/\mathrm{Diff}(L).$$

In standard situations (for example when L is compact) this actually defines a local homeomorphism between the natural topologies on these spaces.

A foundation for the theory of Lagrangian neighbourhoods is provided by the following linear-algebraic construction. Let W be a finite-dimensional real vector space. Then $W \oplus W^*$ admits a canonical symplectic structure $\hat{\omega}$ defined as follows:

$$\hat{\omega}(w_1+\alpha_1, w_2+\alpha_2) := \alpha_2(w_1) - \alpha_1(w_2). \tag{4.1}$$

It turns out that this example of symplectic vector space is actually very general, in the following sense. Let (V,ω) be a symplectic vector space. Let $W \subset V$ be a Lagrangian subspace. Choose a Lagrangian complement $Z \subset V$, so that $V = W \oplus Z$. It is simple to check that the restriction of ω to Z defines an isomorphism

$$\omega_{|Z} : Z \to W^*, \quad z \mapsto \omega(z,\cdot) \tag{4.2}$$

and that, using this isomorphism, one can build an isomorphism $\gamma : (W \oplus W^*, \hat{\omega}) \simeq (V,\omega)$. Furthermore, such γ is unique if we impose that it coincide with the identity on W. Adding this condition thus implies that γ is uniquely defined by the choice of Z.

It is a well-known fact, first noticed by Souriau [22], that a similar construction exists also for symplectic manifolds. The construction is based on the following standard facts. Given any manifold L, the cotangent bundle T^*L admits a canonical symplectic structure $\hat{\omega}$. Specifically, consider the *tautological* 1-form on T^*L defined by $\hat{\lambda}[\alpha](v) := \alpha(\pi_*(v))$, where $\pi : T^*L \to L$ is the natural projection. Then $\hat{\omega} := -d\hat{\lambda}$. Notice that a section of T^*L is simply a 1-form α on L. The graph $\Gamma(\alpha)$ is Lagrangian in T^*L iff α is closed. In particular the zero section $L \subset T^*L$ is Lagrangian. Furthermore each fibre $\pi^{-1}(p) = T^*_pL$ is a Lagrangian submanifold. The fibres thus define a Lagrangian foliation of T^*L transverse to the zero section. Finally, every 1-form α defines a *translation* map

$$\tau_\alpha : T^*L \to T^*L, \quad \tau_\alpha(x,\eta) := (x, \alpha(x)+\eta). \tag{4.3}$$

If α is closed then this map is a symplectomorphism of $(T^*L, \hat{\omega})$.

4.1. **First case: smooth compact Lagrangian submanifolds.** We can now quote Souriau's result, following Weinstein [24] Corollary 6.2.

Theorem 4.2. *Let (M,ω) be a symplectic manifold. Let $L \subset M$ be a smooth compact Lagrangian submanifold. Then there exist a neighbourhood $\mathcal{U}$ of the zero section of L inside its cotangent bundle T^*L and an embedding $\Phi_L : \mathcal{U} \to M$ such that $\Phi_{L|L} = Id : L \to L$ and $\Phi_L^*\omega = \hat{\omega}$.*

Proof. For each $x \in L$, T_xL is a Lagrangian subspace of T_xM. The first step is to choose a Lagrangian complement Z_x, so that $T_xM = T_xL \oplus Z_x$. This can be done smoothly with respect to x using the fact that the space of Lagrangian complements is a contractible set inside the Grassmannian of m-planes in T_xM. As seen following Equation 4.2, ω then provides an isomorphism $\gamma_x : (T_xL \oplus T^*_xL, \hat{\omega}) \to (T_xM, \omega)$, uniquely defined by the condition that $\gamma_x = Id$ on T_xL. Now choose a diffeomorphism $\Psi_L : \mathcal{U} \to M$ such that $(\Psi_L)_*$ extends γ. By construction, the pull-back form $(\Psi_L)^*\omega$ coincides with $\hat{\omega}$ at each point of L. We now need to perturb Ψ_L so that the pull-back form coincides with $\hat{\omega}$ in a neighbourhood of L. Set $\omega_0 := \hat{\omega}$ and $\omega_1 := (\Psi_L)^*\omega$. One can use an argument due to Moser together with the Poincaré Lemma to prove that there exists a diffeomorphism $k : T^*L \to T^*L$ such that $k^*\omega_1 = \omega_0$ and $k_{|L} = Id : L \to L$. Thus $\Phi_L := \Psi_L \circ k$ has the required properties. For later use it is also useful

to note that, using the same argument as in [24] Theorem 7.1, one can further show that, at each $x \in L$, k_* preserves $T^*_x L$. A linear-algebraic argument then shows that this implies that $k_* = Id$ at each $x \in L$. Thus $(\Phi_L)_* = (\Psi_L)_*$ at each $x \in L$. □

Remark 4.3. Although the statement and proof are for embedded submanifolds it is not difficult to extend them to immersed compact Lagrangian submanifolds by working locally. In this case Φ_L will only be a local embedding.

Let $C^\infty(\mathcal{U})$ denote the space of smooth 1-forms on L whose graph lies in $\mathcal{U}$. Theorem 4.2 leads immediately to the following conclusion.

Corollary 4.4. *Let (M, ω) be a symplectic manifold. Let $L \subset M$ be a smooth compact Lagrangian submanifold. Then Φ_L defines by composition an injective map*

$$\Phi_L : C^\infty(\mathcal{U}) \to Imm(L, M)/Diff(L). \tag{4.4}$$

A section $\alpha \in C^\infty(\mathcal{U})$ is closed iff the corresponding (non-parametrized) submanifold $\Phi_L \circ \alpha$ is Lagrangian.

An important point about the map Φ_L in Equation 4.4 is that any submanifold which admits a parametrization which is C^1-close to some parametrization of L belongs to the image of Φ_L, *i.e.* corresponds to a 1-form α.

Let $\mathrm{Lag}(L, M)$ denote the set of Lagrangian immersions from L into M. Using Corollary 4.4 and the Fréchet topology on $C^\infty(\mathcal{U})$ we can locally define a topology on $\mathrm{Lag}(L, M)/\mathrm{Diff}(L)$; one can then check that on the intersection of any two open sets these topologies coincide, so we obtain a global topology on $\mathrm{Lag}(L, M)/\mathrm{Diff}(L)$. The connected component containing the given $L \subset M$ defines the *moduli space of Lagrangian deformations of* L. Coupling Corollary 4.4 with Decomposition 1 gives a good idea of the local structure of this space.

4.2. **Second case: Lagrangian cones in $\mathbb{C}^m$.** Let $\mathcal{C}$ be a Lagrangian cone in $\mathbb{C}^m$ with link (Σ, g') and conical metric $\tilde{g}$. The goal of this section is to provide an analogue of the theory of Section 4.1 for this specific submanifold, giving a correspondence between closed 1-forms in $C^\infty_{(\mu-1,\lambda-1)}(\Lambda^1)$ and Lagrangian deformations of $\mathcal{C}$ with rate (μ, λ).

Let θ denote the generic point on Σ. We will identify $\Sigma \times (0, \infty)$ with $\mathcal{C}$ via the immersion

$$\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m, \quad (\theta, r) \mapsto r\theta. \tag{4.5}$$

Remark 4.5. Let $\theta(t)$ be a curve in Σ such that $\theta(0) = \theta$. Let $r(t)$ be a curve in $\mathbb{R}^+$ such that $r(0) = r$. Differentiating ι at the point (θ, r) gives identifications

$$\begin{array}{rcl} \iota_* : T_\theta\Sigma \oplus \mathbb{R} & \to & T_{r\theta}\mathcal{C} \subset \mathbb{C}^m \\ (\theta'(0), r'(0)) & \mapsto & d/dt\,(r(t)\theta(t))_{|t=0} = r'(0)\theta + r\theta'(0) \in \mathbb{C}^m. \end{array} \tag{4.6}$$

This leads to the general formula $\iota_{*|\theta,r}(v, a) = a\theta + rv$.

We can build an explicit (local) identification $\Psi_{\mathcal{C}}$ of $T^*\mathcal{C}$ with $\mathbb{C}^m$ as follows.

Firstly, the metric $\tilde{g}$ gives an identification

$$T^*\mathcal{C} \to T\mathcal{C}, \quad (\theta, r, \alpha_1 + \alpha_2\, dr) \mapsto (\theta, r, r^{-2}A_1 + \alpha_2 \partial r), \tag{4.7}$$

where $g'(A_1, \cdot) = \alpha_1$ and we use the notation of Section 2.2. Notice that, according to Remark 4.5, the corresponding vector in $\mathbb{C}^m$ is $\iota_*(r^{-2}A_1 + \alpha_2\partial r) = \alpha_2\theta + r^{-1}A_1$. Notice also that Equation 4.7 defines a fibrewise isometry between vector bundles over $\mathcal{C}$. Let $\widetilde{\nabla}$ denote the standard connection on the tangent bundle of $\mathbb{C}^m$. Since $\mathcal{C}$ has the induced metric, the Levi-Civita connection on $T\mathcal{C}$ coincides with the tangential projection $\widetilde{\nabla}^T$. Let $T^*\mathcal{C}$ have the induced Levi-Civita connection. Then Equation 4.7 also defines an isomorphism between the two connections.

Secondly, since $\mathcal{C}$ is Lagrangian the complex structure provides an identification

$$\tilde{J} : T\mathcal{C} \simeq N\mathcal{C}. \tag{4.8}$$

This is again a fibrewise isometry. The perpendicular component $\widetilde{\nabla}^\perp$ defines a connection on $N\mathcal{C}$. Since $\mathbb{C}^m$ is Kähler, $\widetilde{\nabla}\tilde{J} = \tilde{J}\widetilde{\nabla}$. Thus $\widetilde{\nabla}^\perp\tilde{J} = \tilde{J}\widetilde{\nabla}^T$, so Equation 4.8 defines an isomorphism between the two connections.

Thirdly, the Riemannian tubular neighbourhood theorem gives an explicit (local) identification

$$N\mathcal{C} \to \mathbb{C}^m, \quad v \in N_{r\theta}\mathcal{C} \mapsto r\theta + v. \tag{4.9}$$

By composition we now obtain the required identification

$$\Psi_{\mathcal{C}} : \mathcal{U} \subset T^*\mathcal{C} \to \mathbb{C}^m, \quad (\theta, r, \alpha_1 + \alpha_2\, dr) \mapsto r\theta + \tilde{J}(\alpha_2\theta + r^{-1}A_1). \tag{4.10}$$

Now let α be a 1-form on $\mathcal{C}$. Then, under the above identifications, $(\Psi_{\mathcal{C}} \circ \alpha) - \iota \simeq \alpha$. This shows that if $\alpha \in C^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$ for some $\mu > 2$, $\lambda < 2$ then $\Psi_{\mathcal{C}} \circ \alpha$ is a CS/AC submanifold in $\mathbb{C}^m$ asymptotic to $\mathcal{C}$ with rate (μ, λ).

Notice also that

$$\Psi_{\mathcal{C}}(\theta, tr, t^2\alpha_1 + t\alpha_2\, dr) = t\Psi_{\mathcal{C}}(\theta, r, \alpha_1 + \alpha_2\, dr). \tag{4.11}$$

This suggests that we define an action of $\mathbb{R}^+$ on $T^*\mathcal{C}$ as follows:

$$\mathbb{R}^+ \times T^*\mathcal{C} \to T^*\mathcal{C}, \quad t \cdot (\theta, r, \alpha_1 + \alpha_2\, dr) := (\theta, tr, t^2\alpha_1 + t\alpha_2\, dr). \tag{4.12}$$

With respect to this action on $T^*\mathcal{C}$ and the standard action by dilations on $\mathbb{C}^m$, Equation 4.11 shows that $\Psi_{\mathcal{C}}$ is an equivariant map.

Remark 4.6. Equation 4.12 introduces an action on $T^*\mathcal{C}$ which rescales both the base space and the fibres. We can also obtain it as follows. On any cotangent bundle T^*L there is a natural action

$$\mathbb{R}^+ \times T^*L \to T^*L, \quad t \cdot (x, \alpha) := (x, t^2\alpha).$$

The induced action on 1-forms is such that, for the tautological 1-form $\hat{\lambda}$, $t^*\hat{\lambda} = t^2\hat{\lambda}$.

When $L = \Sigma \times (0, \infty)$ there is also a natural action

$$\mathbb{R}^+ \times L \to L, \quad t \cdot (\theta, r) := (\theta, tr).$$

This induces an action on T^*L as follows:

$$\mathbb{R}^+ \times T^*(\Sigma \times (0,\infty)) \to T^*(\Sigma \times (0,\infty)), \quad t \cdot (\theta, r, \alpha_1 + \alpha_2\, dr) := (\theta, tr, \alpha_1 + t^{-1}\alpha_2\, dr).$$

The induced action on 1-forms preserves $\hat{\lambda} : t^*\hat{\lambda} = \hat{\lambda}$. Equation 4.12 coincides with the composed action and thus satisfies $t^*\hat{\lambda} = t^2\hat{\lambda}$, so $t^*\hat{\omega} = t^2\hat{\omega}$.

We now want to investigate the symplectic properties of the map $\Psi_{\mathcal{C}}$. Let $\tilde{\omega}$ denote the standard symplectic structure on $\mathbb{C}^m$. Since $\mathcal{C}$ is Lagrangian, the fibres of the normal bundle define (locally) a Lagrangian foliation of $\mathbb{C}^m$, transverse to $\mathcal{C}$. Using the fact that $\Psi_{\mathcal{C}}$ is the identity on $\mathcal{C}$ and is linear on each fibre, one can check that, at each point of $\mathcal{C}$, $(\Psi_{\mathcal{C}})^*\tilde{\omega} = \hat{\omega}$. Notice also that $\Psi_{\mathcal{C}}$ identifies the foliation of $\mathbb{C}^m$ with the foliation of $T^*\mathcal{C}$ defined by the fibres.

As in the proof of Theorem 4.2, we now want to perturb $\Psi_{\mathcal{C}}$ so as to obtain a local symplectomorphism $\mathcal{U} \subset T^*\mathcal{C} \to \mathbb{C}^m$. As in that case, the idea is to build a (local) diffeomorphism $k : T^*\mathcal{C} \to T^*\mathcal{C}$ such that $k_* = Id$ at each point of $\mathcal{C}$ and $k^*(\Psi_{\mathcal{C}})^*\tilde{\omega} = \hat{\omega}$. The construction of such k is sufficiently explicit in [24] p. 333 to allow us to prove that k is equivariant with respect to the $\mathbb{R}^+$-action. Furthermore, the fact that the fibres of $T^*\mathcal{C}$ are Lagrangian for both symplectic forms implies that k preserves these fibres, see [24] Theorem 7.1 for details. Now define

$$\Phi_{\mathcal{C}} := \Psi_{\mathcal{C}} \circ k : \mathcal{U} \subset T^*\mathcal{C} \to \mathbb{C}^m. \tag{4.13}$$

By construction, $\Phi_{\mathcal{C}}$ satisfies $(\Phi_{\mathcal{C}})^*\tilde{\omega} = \hat{\omega}$. Furthermore, $\Phi_{\mathcal{C}}$ is equivariant and its fibrewise linearization at each $x \in \mathcal{C}$ coincides with $\Psi_{\mathcal{C}}$. Thus $\Phi_{\mathcal{C}} = \Psi_{\mathcal{C}} + R$, for some R satisfying

$$|R(\theta, 1, \alpha_1, \alpha_2)| = O(|\alpha_1|^2_{g'} + |\alpha_2|^2), \quad \text{as } |a_1|_{g'} + |a_2| \to 0. \tag{4.14}$$

Clearly R is also equivariant. Thus

$$|R(\theta, t, \alpha_1, \alpha_2)| = |R(\theta, t \cdot 1, t^2 t^{-2}\alpha_1, t t^{-1}\alpha_2)| = t \cdot O(t^{-4}|\alpha_1|^2_{g'} + t^{-2}|\alpha_2|^2). \tag{4.15}$$

The equivariance of R can be used to determine its asymptotic behaviour with respect to r after composition with 1-forms on $\mathcal{C}$. For example, given any $\mu > 2$ and $\lambda < 2$, choose α in the space $C^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$. Notice that, as $r \to \infty$, $r^{-1}|\alpha_1|_{g'} = |\alpha_1|_g = O(r^{\lambda-1})$. This implies $r^{-4}|\alpha_1|^2_{g'} = O(r^{2\lambda-4})$. Analogously, $|\alpha_2| = O(r^{\lambda-1})$ so $r^{-2}|\alpha_2|^2 = O(r^{2\lambda-4})$. Equation 4.15 then shows that $(R \circ \alpha)(\theta, r) = R(\theta, r, \alpha(\theta, r))$ satisfies $|R \circ \alpha| = O(r^{2\lambda-3})$ as $r \to \infty$. Further calculations show that the derivatives of $R \circ \alpha$ scale correspondingly, *e.g.*

$$|(R \circ \alpha)_*(\partial r)| = O(r^{2\lambda-4}), \quad |(R \circ \alpha)_*(r^{-1}\partial\theta_i)| = O(r^{2\lambda-4}). \tag{4.16}$$

More generally, $|\widetilde{\nabla}^k(R \circ \alpha)| = O(r^{2\lambda-3-k})$. As a result,

$$\begin{aligned} |\widetilde{\nabla}^k(\Phi_{\mathcal{C}} \circ \alpha - \iota)| &= |\widetilde{\nabla}^k((\Psi_{\mathcal{C}} + R) \circ \alpha - \iota)| \leq |\widetilde{\nabla}^k(\Psi_{\mathcal{C}} \circ \alpha - \iota)| + |\widetilde{\nabla}^k(R \circ \alpha)| \\ &= O(r^{\lambda-1-k}) + O(r^{2\lambda-3-k}) = O(r^{\lambda-1-k}), \end{aligned}$$

where we use $\lambda < 2$. This shows that $\Phi_{\mathcal{C}} \circ \alpha$ is a CS/AC Lagrangian submanifold asymptotic to $\mathcal{C}$ with rate (μ, λ). Conversely, one can show that any Lagrangian submanifold L of $\mathbb{C}^m$ which admits a parametrization which is C^1-close to ι and which is asymptotic to ι in the sense of Equations 3.1 and 3.2 corresponds to a closed 1-form $\alpha \in C^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$.

In complete analogy with Section 4.1 we can use $\Phi_{\mathcal{C}}$ and the closed forms in the space $C^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$ to define a topology on the set of Lagrangian submanifolds which admit a parametrization $\iota : \Sigma \times \mathbb{R}^+ \to \mathbb{C}^m$ which is asymptotic to $\mathcal{C}$ with rate (μ, λ). The connected component containing $\mathcal{C}$ defines the *moduli space of CS/AC Lagrangian deformations of* $\mathcal{C}$ *with rate* (μ, λ).

We conclude with a last comment on the differential properties of $\Psi_{\mathcal{C}}$. Recall the following general fact.

Lemma 4.7. *Let $E \to M$ be a vector bundle, endowed with a connection ∇. Let $\sigma : M \to E$ be a section of E. Choose $v \in T_pM$. The connection defines a decomposition into "vertical" and "horizontal" subspaces*

$$T_{\sigma(p)}E = V_{\sigma(p)} \oplus H_{\sigma(p)}, \quad \textit{with} \;\; V_{\sigma(p)} \simeq E_p, \;\; H_{\sigma(p)} \simeq T_p(M). \tag{4.17}$$

Under these identifications, $\sigma_*(v) \simeq \nabla_v\sigma + v$.

We can apply Lemma 4.7 as follows. Let α be a section of $T^*\mathcal{C}$ so that $\Psi_{\mathcal{C}} \circ \alpha : \mathcal{C} \to \mathbb{C}^m$ is a submanifold of $\mathbb{C}^m$. Choose $v \in T_{r\theta}\mathcal{C}$. Then, using the identifications 4.7, 4.8, 4.9 and Lemma 4.7,

$$(\Psi_{\mathcal{C}} \circ \alpha)_*(v) \simeq \widetilde{\nabla}_v\alpha + v, \tag{4.18}$$

where $\widetilde{\nabla}$ denotes the Levi-Civita connection on $T^*\mathcal{C}$.

4.3. **Third case: CS/AC Lagrangian submanifolds in $\mathbb{C}^m$.** Let $\iota : L \to \mathbb{C}^m$ be an AC Lagrangian submanifold with rate $\boldsymbol{\lambda}$, centers p_i and ends S_i. Using the notation of Section 4.2, the map $\Phi_{\mathcal{C}_i} + p_i : T^*\mathcal{C}_i \to \mathbb{C}^m$ identifies $\iota(S_i) \subset \mathbb{C}^m$ with the graph $\Gamma(\alpha_i)$ of some closed 1-form α_i. This construction also determines a distinguished coordinate system ϕ_i by imposing the relation

$$\phi_i : \mathcal{C}_i \to S_i, \quad \iota \circ \phi_i = \Phi_{\mathcal{C}_i} \circ \alpha_i.$$

Letting $(d\phi_i)^* : T^*S_i \to T^*\mathcal{C}_i$ denote the corresponding identification of cotangent bundles, we obtain an identification of the zero section $\mathcal{C}_i$ with the zero section S_i. We can use the symplectomorphism τ_{α_i} defined in Equation 4.3 to "bridge the gap" between these identifications, obtaining a symplectomorphism

$$\Phi_{S_i} : \mathcal{U}_i \subset T^*S_i \to \mathbb{C}^m, \quad \Phi_{S_i} := \Phi_{\mathcal{C}_i} \circ \tau_{\alpha_i} \circ (d\phi_i)^* + p_i \tag{4.19}$$

which restricts to the identity on S_i. These maps provide a Lagrangian neighbourhood for each end of L. Using the same methods as in the proof of Theorem 4.2 one can interpolate between these maps. The final result is a symplectomorphism

$$\Phi_L : \mathcal{U} \subset T^*L \to \mathbb{C}^m \tag{4.20}$$

which restricts to the identity along L. This allows us to parametrize AC deformations of L with rate $\boldsymbol{\lambda}$ in terms of closed 1-forms in the space $C^\infty_{\boldsymbol{\lambda}-1}(\mathcal{U})$.

More generally, given a CS or CS/AC Lagrangian submanifold L in $\mathbb{C}^m$, the same ideas define a symplectomorphism Φ_L as in Equation 4.20. The same is true for a CS submanifold in M: this time it is necessary to insert appropriate compositions by Υ_i. We refer to Joyce [10] for additional details concerning constructions of this type.

Coupling these results with Decompositions 2, 3 and 4 now gives a good idea of the local structure of the corresponding moduli spaces of Lagrangian deformations, defined as in Sections 4.1 and 4.2.

4.4. **Lagrangian deformations with moving singularities.** In Section 4.3 the given Lagrangian submanifold L is deformed keeping the singular points fixed in the ambient manifold $\mathbb{C}^m$ or M. It is also natural to want to deform L allowing the singular points to move within the ambient space. Analogously, one might want to allow the corresponding Lagrangian cones $\mathcal{C}_i$ to rotate in $\mathbb{C}^m$. The correct set-up for doing this when $\iota : L \to M$ is a CS Lagrangian submanifold with singularities $\{x_1, \dots, x_s\}$ and identifications υ_i is as follows. The ideas are based on [11] Section 5.1. Define

$$P := \{(p, \upsilon) : p \in M,\ \upsilon : \mathbb{C}^m \to T_pM \text{ such that } \upsilon^*\omega = \tilde{\omega}, \upsilon^*J = \tilde{J}\}. \tag{4.21}$$

P is a U(m)-principal fibre bundle over M with the action

$$\mathrm{U(m)} \times P \to P, \quad M \cdot (p, \upsilon) := (p, \upsilon \circ M^{-1}).$$

As such, P is a smooth manifold of dimension $m^2 + 2m$.

Our aim is to use one copy of P to parametrize the location of each singular point $p_i = \iota(x_i) \in M$ and the direction of the corresponding cone $\mathcal{C}_i \subset \mathbb{C}^m$: the group action will allow the cone to rotate leaving the singular point fixed. As we are interested only in small deformations of L we can restrict our attention to a small open neighbourhood of the pair $(p_i, \upsilon_i) \in P$. In general the $\mathcal{C}_i$ will have some symmetry group $G_i \subset \mathrm{U(m)}$, *i.e.* the action of this G_i will leave the cone fixed. To ensure that we have no redundant parameters we must therefore further restrict our attention to a *slice* of our open neighbourhood, *i.e.* a smooth submanifold transverse to the orbits of G_i. We denote this slice $\mathcal{E}_i$: it is a subset of P containing (p_i, υ_i) and of dimension $m^2 + 2m - \dim(G_i)$. We then set $\mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_s$. The point $e := (p_1, \upsilon_1), \dots, (p_s, \upsilon_s)) \in \mathcal{E}$ will denote the initial data as in Definition 3.6.

We now want to extend the datum of (L, ι) to a family of Lagrangian submanifolds $(L, \iota_{\tilde{e}})$ parametrized by $\tilde{e} = ((\tilde{p}_1, \tilde{\upsilon}_1), \dots, (\tilde{p}_s, \tilde{\upsilon}_s)) \in \mathcal{E}$ (making $\mathcal{E}$ smaller if necessary). Each $(L, \iota_{\tilde{e}})$ should satisfy $\iota_{\tilde{e}}(p_i) = \tilde{p}_i$ and admit identifications $\tilde{\upsilon}_i$ and cones $\mathcal{C}_i$ as in Definition 3.6. We further require that $\iota_e = \iota$ globally and that $\iota_{\tilde{e}} = \iota$ outside a neighbourhood of the singularities. The construction of such a family is actually straight-forward: using the maps Υ_i, it reduces to a choice of an appropriate family of compactly-supported symplectomorphisms of $\mathbb{C}^m$.

It is now possible to choose an open neighbourhood $\mathcal{U} \subset T^*L$ and embeddings $\Phi_L^{\tilde{e}} : \mathcal{U} \to M$ which, away from the singularities, coincide with the embedding Φ_L introduced in Section 4.3. The final result is that, after such a choice, the *moduli space of CS Lagrangian deformations of L with rate $\boldsymbol{\mu}$ and moving singularities* can be parametrized in terms of pairs $(\tilde{e}, \alpha)$ where $\tilde{e} \in \mathcal{E}$ and α is a closed 1-form on L belonging to the space $C^\infty_{\boldsymbol{\mu}-1}(\mathcal{U})$.

Analogous results hold of course for CS and CS/AC submanifolds in $\mathbb{C}^m$. In this case it is sufficient to set $P := \{(p, \upsilon)\}$, with $p \in \mathbb{C}^m$ and $\upsilon \in \mathrm{U(m)}$.

4.5. **Other convergence rates.** The previous sections discuss the deformation theory of a Lagrangian conifold (L, ι) with convergence rate $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ within the class of deformations which preserve the convergence rate. For some purposes, cf. [21], it may also be useful to consider other deformation classes, obtained via closed 1-forms in the space $C^\infty_{\boldsymbol{\beta}-1}(\mathcal{U})$, for some other weight $\boldsymbol{\beta}$. We consider here two cases.

The first case is when $2 < \beta_i \leq \mu_i$ on each CS end and $2 > \beta_i \geq \lambda_i$ on each AC end: in other words, we relax the convergence rate of the deformed submanifolds. This case is simple: the initial conifold has *a fortiori* convergence rate $\boldsymbol{\beta}$, so the above theory immediately shows that the closed forms in $C^\infty_{\boldsymbol{\beta}-1}(\mathcal{U})$ parametrize all other Lagrangian conifolds with this rate.

The second case is when $2 < \mu_i < \beta_i$ on each CS end and $2 > \lambda_i > \beta_i$ on each AC end: in other words, we strengthen the convergence rate. In this case the closed 1-forms in $C^\infty_{\boldsymbol{\beta}-1}(\mathcal{U})$ parametrize the Lagrangian immersions (L, ι') which are asymptotic to (L, ι) in a sense analogous to Definitions 3.2, 3.3: on each AC end, up to appropriate diffeomorphisms ϕ_i and for $r \to \infty$,

$$|\widetilde{\nabla}^k(\iota' - \iota)| = O(r^{\beta_i - 1 - k})$$

and on each CS end, for $r \to 0$,

$$|\widetilde{\nabla}^k(\iota' - \iota)| = O(r^{\beta_i - 1 - k}).$$

To prove this, as in Section 4.3, assume ι is obtained as the graph of a 1-form α so that on each end $\iota = \Phi_{\mathcal{C}} \circ \alpha$ and $\Phi_L = \Phi_{\mathcal{C}} \circ \tau_\alpha$. Choose a closed 1-form $\alpha' \in C^\infty_{\boldsymbol{\beta}-1}(\mathcal{U})$ and set $\iota' := \Phi_L \circ \alpha' = \Phi_{\mathcal{C}} \circ (\alpha + \alpha')$. Then

$$\begin{aligned} |\iota' - \iota| &= |\Phi_{\mathcal{C}} \circ (\alpha + \alpha') - \Phi_{\mathcal{C}} \circ \alpha| = |(\Psi_{\mathcal{C}} + R) \circ (\alpha + \alpha') - (\Psi_{\mathcal{C}} + R) \circ \alpha| \\ &\leq |\Psi_{\mathcal{C}} \circ \alpha'| + |R \circ (\alpha + \alpha') - R \circ \alpha| \\ &= O(r^{\beta - 1}) + O(r^{\beta - 2 + \lambda - 2 + 1}) = O(r^{\beta - 1}), \end{aligned}$$

where we use the fact that $R(\theta, 1, \cdot)$ is roughly quadratic in the $\cdot$ variable, so

$$|R(\theta, 1, \alpha' + \alpha) - R(\theta, 1, \alpha)| = O(|\alpha'| \cdot |\alpha|).$$

We then conclude via the reasoning already described following Equation 4.15.

Similar calculations give estimates on the derivatives.

5. Special Lagrangian conifolds

Definition 5.1. A *Calabi-Yau* (CY) manifold is the data of a Kähler manifold (M^{2m}, g, J, ω) and a non-zero $(m, 0)$-form Ω satisfying $\nabla \Omega \equiv 0$ and normalized by the condition $\omega^m / m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}$.

In particular Ω is holomorphic and the holonomy of (M, g) is contained in SU(m). We will refer to Ω as the *holomorphic volume form* on M.

Example 5.2. The simplest example of a CY manifold is $\mathbb{C}^m$ with its standard structures $\tilde{g}$, $\tilde{J}$, $\tilde{\omega}$ and $\tilde{\Omega} := dz^1 \wedge \cdots \wedge dz^m$.

Definition 5.3. Let M^{2m} be a CY manifold and $L^m \to M$ be an immersed or embedded Lagrangian submanifold. We can restrict Ω to L, obtaining a non-vanishing complex-valued m-form $\Omega_{|L}$ on L. We say that L is *special Lagrangian* (SL) iff this form is real, *i.e.* Im $\Omega_{|L} \equiv 0$. In this case Re $\Omega_{|L}$ defines a volume form on L, thus a natural orientation.

Lagrangian submanifolds (especially the immersed ones) tend to be very "soft" objects: for example, Section 4 shows that they have infinite-dimensional moduli spaces. They also easily allow for cutting, pasting and desingularization procedures. The "special" condition rigidifies them considerably: the corresponding deformation, gluing and desingularization processes require much "harder" techniques. Cf. *e.g.* [5], [12], [13], [21] for recent gluing results and [6] for local desingularization issues.

Definition 5.4. We can define AC, CS and CS/AC special Lagrangian submanifolds in $\mathbb{C}^m$ exactly as in Definitions 3.2, 3.3 and 3.4, simply adding the requirement that the submanifolds be special Lagrangian. In particular this implies that the cones $\mathcal{C}_i$ are SL in $\mathbb{C}^m$. Following Definition 3.6 we can also define CS special Lagrangian submanifolds in a general CY manifold M: in this case it is necessary to also add the requirement that $v_i^*\Omega = \tilde{\Omega}$.

We use the generic term *special Lagrangian conifold* to refer to any of the above.

Remark 5.5. It follows from Joyce [10] Theorem 5.5 that if L is a CS or CS/AC SL submanifold with respect to some rate $\boldsymbol{\mu} = 2+\epsilon$ with ϵ in a certain range $(0, \epsilon_0)$ then it is also CS or CS/AC with respect to any other rate of the form $\boldsymbol{\mu}' = 2 + \epsilon'$ with $\epsilon' \in (0, \epsilon_0)$. The precise value of ϵ_0 is determined by certain *exceptional weights* for the cones $\mathcal{C}_i$, introduced in Section 7. We refer to [10] for details.

Example 5.6. Let $\mathcal{C}$ be a Lagrangian cone in $\mathbb{C}^m$ with smooth link Σ^{m-1}. It can be shown that $\mathcal{C}$ is SL (with respect to some holomorphic volume form $e^{i\theta}\tilde{\Omega}$) iff Σ is minimal in $\mathbb{S}^{2m-1}$ with respect to the natural metric on the sphere. Then $\mathcal{C}$ is a CS/AC SL in $\mathbb{C}^m$. Cf. *e.g.* [3], [4], [5], [6], [8] for examples.

We refer to Joyce [9] Section 6.4 for examples of AC SLs in $\mathbb{C}^m$ with various rates.

6. Setting up the SL deformation problem

If $\iota : L \to M$ is a SL conifold we can specialize the framework of Section 4 to study the SL deformations of L. Notice that the SL condition is again invariant under reparametrizations. Thus, if L is smooth and compact, the *moduli space* $\mathcal{M}_L$ *of SL deformations of* L can be defined as the connected component containing L of the subset of SL submanifolds in $\mathrm{Lag}(L, M)/\mathrm{Diff}(L)$. As seen in Sections 4.2 and 4.3, if L is an AC, CS or CS/AC Lagrangian submanifold with specific rates of growth/decay on the ends, we can obtain moduli spaces of Lagrangian or SL deformations of L with those same rates by simply restricting our attention to closed 1-forms on L which satisfy corresponding growth/decay conditions.

Our ultimate goal is to prove that moduli spaces of SL conifolds often admit a natural smooth structure with respect to which they are finite-dimensional manifolds. Failing this, we want to identify the obstructions which prevent this from happening. Generally speaking, the strategy for proving these results will be to view $\mathcal{M}_L$ locally as the zero set of some smooth map F defined on the space of closed forms in $C^\infty(\mathcal{U})$ (when L is smooth and compact) or in $C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\mathcal{U})$ (when L is CS/AC with rate $(\boldsymbol{\mu}, \boldsymbol{\lambda})$): we can then attempt to use the Implicit Function Theorem to prove that this zero set is smooth.

The choice of F is dictated by Definition 5.3. Let Ω denote the given holomorphic volume form on M. Then F must compute the values of Im Ω on each Lagrangian deformation of L. In the following sections we present the precise construction of F and study its properties, for each case of interest.

Note: To simplify the notation, from now on we will drop the immersion $\iota : L \to M$ and simply identify L with its image. In particular we will identify the singularities x_i with their images $\iota(x_i)$.

6.1. **First case: smooth compact special Lagrangians.** Let $L \subset M$ be a smooth compact SL submanifold, endowed with the induced metric g and orientation. Define $\Phi_L : \mathcal{U} \to M$ as in Section 4.1. Consider the pull-back real m-form $\Phi_L^*(\mathrm{Im}\,\Omega)$ defined on $\mathcal{U}$. Given any closed $\alpha \in C^\infty(\mathcal{U})$, let $\Gamma(\alpha)$ denote the submanifold in $\mathcal{U}$ defined by its graph. It is diffeomorphic to L via the projection $\pi : T^*L \to L$. The pull-back form restricts to an m-form $\Phi_L^*(\mathrm{Im}\,\Omega)_{|\Gamma(\alpha)}$ on $\Gamma(\alpha)$. It is clear from Definition 5.3 that $\Gamma(\alpha)$ is SL iff this form vanishes. We can now pull this form back to L via α (equivalently, push it down to L via π_*), obtaining a real m-form on L: then $\Gamma(\alpha)$ is SL iff this form vanishes on L. Finally, let $\star$ denote the *Hodge star operator* defined on L by g and the orientation. Using this operator we can reduce any m-form on L to a function.

Summarizing, let $\mathcal{D}_L$ denote the space of closed 1-forms on L whose graph lies in $\mathcal{U}$. We then define the map F as follows.

$$F : \mathcal{D}_L \to C^\infty(L), \quad \alpha \mapsto \star(\alpha^*(\Phi_L^*\,\mathrm{Im}\,\Omega)) = \star((\Phi_L \circ \alpha)^*\,\mathrm{Im}\,\Omega). \tag{6.1}$$

Proposition 6.1. *The non-linear map F has the following properties:*

1. *The set $F^{-1}(0)$ parametrizes the space of all SL deformations of L which are C^1-close to L.*
2. *F is a smooth map between Fréchet spaces. Furthermore, for each $\alpha \in \mathcal{D}_L$, $\int_L F(\alpha)\,vol_g = 0$.*
3. *The linearization $dF[0]$ of F at 0 coincides with the operator d^*, i.e.*

$$dF[0](\alpha) = d^*\alpha. \tag{6.2}$$

Proof. These results are standard, cf. [16] or [10] Prop. 2.10. However for the reader's convenience we give a sketch of the argument with respect to our own set of conventions. To simplify the notation we identify $\mathcal{U}$ with its image in M via Φ_L. This allows us to write

$$F(\alpha) = \star(\pi_*(\mathrm{Im}\,\Omega_{|\Gamma(\alpha)})). \tag{6.3}$$

We also identify L with the zero section in T^*L.

The first statement follows directly from the definition of F and the results of Section 4.1. More precisely the statement is that, up to composition with Φ_L, $F^{-1}(0)$ coincides with the set of SL submanifolds which admit a parametrization which is C^1-close to some parametrization of L.

To prove the second statement, notice that $\int_L F(\alpha)\,vol_g = \int_{\Gamma(\alpha)} \mathrm{Im}\,\Omega$. The fact that Ω is closed implies that $\mathrm{Im}\,\Omega$ is closed. Furthermore the submanifold $\Gamma(\alpha)$ is homotopic, thus homologous, to the zero section L. Thus $\int_{\Gamma(\alpha)} \mathrm{Im}\,\Omega = \int_L \mathrm{Im}\,\Omega = 0$ because L is SL. The smoothness of F is clear from its definition.

To prove Equation 6.2, fix any $\alpha \in \Lambda^1(L)$ and let v denote the normal vector field along L determined by imposing $\alpha(\cdot) \equiv \omega(v, \cdot)$. We can extend v to a global vector field v on M. Let ϕ_s denote any 1-parameter family of diffeomorphisms of M such that $d/ds(\phi_s(x))_{|s=0} = v(x)$. Then the two 1-parameter families of m-forms on L, $(s\alpha)^*(\mathrm{Im}\,\Omega) = \pi_*(\mathrm{Im}\,\Omega_{|\Gamma(s\alpha)})$ and $(\phi_s^*\,\mathrm{Im}\,\Omega)_{|L}$, coincide up to first order so that standard calculus of Lie derivatives shows that

$$\begin{aligned} dF[0](\alpha)\,vol_g &= d/ds(F(s\alpha)\,vol_g)_{|s=0} \\ &= d/ds(\phi_s^*\,\mathrm{Im}\,\Omega)_{|L;\,s=0} \\ &= (\mathcal{L}_v\,\mathrm{Im}\,\Omega)_{|L} = (di_v\,\mathrm{Im}\,\Omega)_{|L}, \end{aligned}$$

where in the last equality we use *Cartan's formula* $\mathcal{L}_v = di_v + i_v d$ and the fact that $\mathrm{Im}\,\Omega$ is closed.

We now claim that $(i_v \mathrm{Im}\,\Omega)_{|L} \equiv -\star\alpha$ on L. This is a linear algebra statement so we can check it point by point. We can also assume that v is a unit vector at that point. Fix a point $x \in L$ and an isomorphism $T_xM \simeq \mathbb{C}^m$ identifying the CY structures on T_xM with the standard structures on $\mathbb{C}^m$. This map will identify T_xL with a SL m-plane Π in $\mathbb{C}^m$. Consider the action of SU(m) on the Grassmannian of m-planes in $\mathbb{C}^m$. In [3] page 89 it is shown that SU(m) acts transitively on the subset of SL m-planes and that the isotropy subgroup corresponding to the distinguished SL plane $\mathbb{R}^m := \mathrm{span}\{\partial x^1, \dots, \partial x^m\}$ is SO(m) $\subset$ SU(m); in other words, the set of SL m-planes in $\mathbb{C}^m$ can be identified with the homogeneous space SU(m)/SO(m). Up to a rotation in SU(m) we can assume that $\Pi = \mathbb{R}^m$. Up to a rotation in SO(m) we can further assume that $v(x) = \partial y^1$. It is thus sufficient to check our claim in this case only. We can write $\mathrm{Im}\,\Omega = dy^1 \wedge dx^2 \wedge \dots \wedge dx^m + (\dots)$. It follows that $(i_v \mathrm{Im}\,\Omega)_{|\mathbb{R}^m} = dx^2 \wedge \dots \wedge dx^m$. On the other hand $\alpha = -dx^1$, proving the claim, thus Equation 6.2. □

Remark 6.2. Notice that $\star$ depends on $x \in L$, $\Gamma(\alpha)$ depends on α and $\Phi_L^* \mathrm{Im}\,\Omega_{|\Gamma(\alpha)}$ depends on α and $\nabla\alpha$. We can thus think of F as being obtained from an underlying smooth function

$$F' = F'(x, y, z) : \mathcal{U} \oplus (T^*L \otimes T^*L) \to \mathbb{R} \tag{6.4}$$

via the following relationship:

$$F(\alpha) = F'(x, \alpha(x), \nabla\alpha(x)). \tag{6.5}$$

More specifically, F' can be defined as follows. Choose a point $(x, y) \in \mathcal{U}$. Let $e_1, \dots, e_m$ be an orthonormal positive basis of T_xL. Now choose any $z \in T_x^*L \otimes T_x^*L$. Recall from Lemma 4.7 that, using the Levi-Civita connection, $T_{(x,y)}\mathcal{U} \simeq T_x^*L \oplus T_xL$. Thus the vectors $(i_{e_i}z, e_i)$ span an m-plane in $T_{(x,y)}\mathcal{U}$; when $y = \alpha$ and $z = \nabla\alpha$, this m-plane coincides with $T_{(x,\alpha)}\Gamma(\alpha)$. We can now define

$$F'(x, y, z) := \Phi_L^* \mathrm{Im}\,\Omega_{|(x,y)}((i_{e_1}z, e_1), \dots, (i_{e_m}z, e_m)). \tag{6.6}$$

For any fixed $x \in L$, y and z vary in the linear space $T_x^*L \oplus (T_x^*L \otimes T_x^*L)$ so Taylor's theorem shows

$$F'(x, y, z) = F'(x, 0, 0) + \frac{\partial F'}{\partial y}(x, 0, 0)\, y + \frac{\partial F'}{\partial z}(x, 0, 0)\, z + Q'(x, y, z) \tag{6.7}$$

for some smooth $Q' = Q'(x, y, z)$ satisfying $Q'(x, y, z) = O(|y|^2 + |z|^2)$ for each x, as $|y| \to 0$ and $|z| \to 0$. By substitution we find

$$\begin{aligned} F(\alpha) &= F'(x, \alpha(x), \nabla\alpha(x)) \\ &= F'(x, 0, 0) + \frac{\partial F'}{\partial y}(x, 0, 0)\, \alpha(x) + \frac{\partial F'}{\partial z}(x, 0, 0)\, \nabla\alpha(x) + Q'(x, \alpha(x), \nabla\alpha(x)). \end{aligned}$$

The fact that L is SL implies that $F'(x, 0, 0) \equiv 0$. Notice also that by the chain rule

$$d/ds(F(s\alpha))_{|s=0} = d/ds(F'(x, s\alpha(x), s\nabla\alpha(x))_{|s=0} = \frac{\partial F'}{\partial y}(x, 0, 0)\, \alpha(x) + \frac{\partial F'}{\partial z}(x, 0, 0)\, \nabla\alpha(x).$$

On the other hand, $d/ds(F(s\alpha))_{|s=0} = dF[0](\alpha) = d^*\alpha$. Combining these equations leads to

$$F(\alpha) = d^*\alpha + Q'(x, \alpha(x), \nabla\alpha(x)). \tag{6.8}$$

6.2. **Second case: special Lagrangian cones in $\mathbb{C}^m$.** Let $\mathcal{C}$ be a SL cone in $\mathbb{C}^m$, endowed with the induced metric $\tilde{g}$ and orientation. Define $\Phi_{\mathcal{C}}:\mathcal{U}\rightarrow\mathbb{C}^m$ as in Section 4.2. Fix any $\mu>2$, $\lambda<2$. Let $\mathcal{D}_{\mathcal{C}}$ denote the space of closed 1-forms in $C^\infty_{(\mu-1,\lambda-1)}(\Lambda^1)$ whose graph lies in $\mathcal{U}$. Given $\alpha\in\mathcal{D}_{\mathcal{C}}$, define $F(\alpha)$ as in Equation 6.1.

Proposition 6.3. *The non-linear map F has the following properties:*

(1) *The set $F^{-1}(0)$ parametrizes the space of all SL deformations of $\mathcal{C}$ which are C^1-close to L and are asymptotic to $\mathcal{C}$ with rate (μ,λ).*
(2) *F is a well-defined smooth map*

$$F:\mathcal{D}_{\mathcal{C}}\rightarrow C^\infty_{(\mu-2,\lambda-2)}(\mathcal{C}).$$

In particular, for each $\alpha\in\mathcal{D}_{\mathcal{C}}$, $F(\alpha)\in C^\infty_{(\mu-2,\lambda-2)}(\mathcal{C})$.
(3) *The linearization $dF[0]$ of F at 0 coincides with the operator d^*, i.e.*

$$dF[0](\alpha)=d^*\alpha. \tag{6.9}$$

Proof. The first statement follows from the definition of F and the results of Section 4.2. Concerning the second statement, we may write

$$\begin{aligned}F(\alpha) &= \star(\alpha^*(\Phi_{\mathcal{C}}^*\,\mathrm{Im}\,\tilde{\Omega}))=\mathrm{Im}\,\tilde{\Omega}((\Phi_{\mathcal{C}}\circ\alpha)_*(e_1),\dots,(\Phi_{\mathcal{C}}\circ\alpha)_*(e_m))\\ &= \mathrm{Im}\,\tilde{\Omega}((\Psi_{\mathcal{C}}\circ\alpha)_*(e_1)+(R\circ\alpha)_*(e_1),\dots,(\Psi_{\mathcal{C}}\circ\alpha)_*(e_m)+(R\circ\alpha)_*(e_m))\\ &= \mathrm{Im}\,\tilde{\Omega}((\Psi_{\mathcal{C}}\circ\alpha)_*(e_1),\dots,(\Psi_{\mathcal{C}}\circ\alpha)_*(e_m))+\dots,\end{aligned}$$

where e_i is a local $\tilde{g}$-orthornomal basis of $T\mathcal{C}$.

Consider this last equation as $r\rightarrow\infty$. Equation 4.18 shows that its first term is of the form $\mathrm{Im}\,\tilde{\Omega}(e_1,\dots,e_m)+O(r^{\lambda-2})$. The first term here vanishes because $\mathcal{C}$ is SL, leaving the term $O(r^{\lambda-2})$. Equation 4.16 shows that the remaining terms in $F(\alpha)$ are of the form $O(r^{2\lambda-4})$. Analogous methods apply for $r\rightarrow 0$, showing that $F(\alpha)\in C^0_{(\mu-2,\lambda-2)}(\mathcal{C})$.

To study the derivatives of $F(\alpha)$ we endow $\mathcal{U}$ with the metric and Levi-Civita connection ∇ pulled back from $\mathbb{C}^m$ via $\Phi_{\mathcal{C}}$, so that $\nabla(\Phi_{\mathcal{C}}^*\,\mathrm{Im}\,\tilde{\Omega})=\Phi_{\mathcal{C}}^*(\widetilde{\nabla}\,\mathrm{Im}\,\tilde{\Omega})=0$. Let g denote the induced metric on $\Gamma(\alpha)$. Then $\mathcal{C}$ can be endowed with either the metric $\tilde{g}$ and induced connection $\widetilde{\nabla}$ or with the metric α^*g and induced connection ∇. One can check, or cf. [19], that the fact that α^*g is asymptotic to $\tilde{g}$ implies that the difference tensor $A:=\nabla-\widetilde{\nabla}$ satisfies $|A|=O(r^{\lambda-3})$, as $r\rightarrow\infty$. Notice that

$$F(\alpha)\,\mathrm{vol}_{\tilde{g}}=(\Phi_{\mathcal{C}}\circ\alpha)^*\,\mathrm{Im}\,\tilde{\Omega}$$

so, taking derivatives,

$$\nabla(F(\alpha)\,\mathrm{vol}_{\tilde{g}})=\nabla((\Phi_{\mathcal{C}}\circ\alpha)^*\,\mathrm{Im}\,\tilde{\Omega})=(\Phi_{\mathcal{C}}\circ\alpha)^*(\widetilde{\nabla}\,\mathrm{Im}\,\tilde{\Omega})=0.$$

This implies

$$|(\nabla F(\alpha))\otimes\mathrm{vol}_{\tilde{g}}|=|F(\alpha)\cdot\nabla(\mathrm{vol}_{\tilde{g}})|=O(r^{\lambda-2})|\nabla(\mathrm{vol}_{\tilde{g}})|.$$

Write $\mathrm{vol}_{\tilde{g}}=e_1^*\otimes\cdots\otimes e_m^*$ so that $\nabla(\mathrm{vol}_{\tilde{g}})=\nabla e_1^*\otimes\cdots\otimes e_m^*+\cdots+e_1^*\otimes\cdots\otimes\nabla e_m^*$. We may assume that $\widetilde{\nabla}e_i^*=0$. Then $\nabla e_i^*=(\nabla-\widetilde{\nabla})e_i^*=Ae_i^*$, leading to $|\nabla(\mathrm{vol}_{\tilde{g}})|=O(r^{\lambda-3})$. This shows that $F(\alpha)\in C^1_{(\mu-2,\lambda-2)}(\mathcal{C})$. Further calculations of the same type apply to the higher derivatives, showing that $F(\alpha)\in C^\infty_{(\mu-2,\lambda-2)}(\mathcal{C})$. It is clear that F is smooth.

The third statement can be proved as in Proposition 6.1. ∎

6.3. **Third case: CS/AC special Lagrangians in $\mathbb{C}^m$.** Let L be a AC, CS or CS/AC SL in $\mathbb{C}^m$ or a CS SL in M. The moduli space of SL deformations of L with fixed singularities coincides locally with the zero set of a map F defined as in Equation 6.1. The methods and results of Sections 4.3 and 6.2 then lead to a good understanding of the properties of F, analogous to those described in Propositions 6.1 and 6.3. For example, assume L is a CS SL in M with rate $\boldsymbol{\mu}$. Let $\mathcal{D}_L$ denote the space of closed 1-forms in $C^\infty_{\boldsymbol{\mu}-1}(\Lambda^1)$ whose graph lies in $\mathcal{U}$. Choose $\alpha \in \mathcal{D}_L$. Then one can check that $F(\alpha) \in C^\infty_{\boldsymbol{\mu}-2}(L)$. The calculation is similar to the one already used in the proof of Proposition 6.3. In particular it uses (i) the fact that the asymptotic cones $\mathcal{C}_i$ are SL, (ii) the fact that the discrepancy between the forms Ω and $\tilde{\Omega}$ is of the order $O(r) < O(r^{\mu_i - 2})$.

We now want to understand how to parametrize the SL deformations of L whose singularities are allowed to move in the ambient space as in Section 4.4. For example, assume L is a CS SL submanifold in M. The constructions of Section 4.4 must then be modified as follows. This time we set

$$\tilde{P} := \{(p, \upsilon) : p \in M,\ \upsilon : \mathbb{C}^m \to T_pM \text{ such that } \upsilon^*\omega = \tilde{\omega},\ \upsilon^*\Omega = \tilde{\Omega}\}, \tag{6.10}$$

so that $\tilde{P}$ is a SU(m)-principal fibre bundle over M of dimension m^2+2m-1. For each end, the cone $\mathcal{C}_i$ will now have symmetry group $G_i \subset \mathrm{SU(m)}$. As in Section 4.4, let $\tilde{\mathcal{E}}_i$ denote a smooth submanifold of $\tilde{P}$ transverse to the orbits of G_i. It has dimension $m^2 + 2m - 1 - \dim(G_i)$. Set $\tilde{\mathcal{E}} := \tilde{\mathcal{E}}_1 \times \cdots \times \tilde{\mathcal{E}}_s$. We then define CS Lagrangian submanifolds $L_{\tilde{e}}$ and embeddings $\Phi_L^{\tilde{e}}$ with the same properties as before.

Now let $\mathcal{D}_L$ denote the space of closed 1-forms in $C^\infty_{\boldsymbol{\mu}-1}(\Lambda^1)$ whose graph lies in $\mathcal{U}$. We define a map

$$F : \tilde{\mathcal{E}} \times \mathcal{D}_L \to C^\infty_{\boldsymbol{\mu}-2}(L), \quad (\tilde{e}, \alpha) \mapsto \star(\alpha^*(\Phi_L^{\tilde{e}*} \operatorname{Im} \Omega)). \tag{6.11}$$

Proposition 6.4. *Let L be a CS SL in M. Then the map F has the following properties:*

1. *The set $F^{-1}(0)$ parametrizes the space of all SL deformations of L which are C^1-close to L away from the singularities and are asymptotic to $\mathcal{C}_i$ with rate μ_i for some choice of $(\tilde{p}_i, \tilde{\upsilon}_i)$ near (p_i, υ_i).*
2. *F is a (locally) well-defined smooth map between Fréchet spaces. In particular, for each $\alpha \in \mathcal{D}_L$, $F(\alpha) \in C^\infty_{\boldsymbol{\mu}-2}(L)$. Furthermore, $\int_L F(\alpha)\, vol_g = 0$.*
3. *There exists an injective linear map $\chi : T_e\tilde{\mathcal{E}} \to C^\infty_{\mathbf{0}}(L)$ such that (i) $\chi(y) \equiv 0$ away from the singularities and (ii) the linearized map $dF[0] : T_e\tilde{\mathcal{E}} \oplus C^\infty_{\boldsymbol{\mu}-1}(\Lambda^1) \to C^\infty_{\boldsymbol{\mu}-2}(L)$ satisfies*

$$dF[0](y, \alpha) = \Delta_g\, \chi(y) + d^*\alpha. \tag{6.12}$$

Proof. The first statement should be interpreted as explained in the proof of Proposition 6.1. The proof follows from the definitions of $\tilde{\mathcal{E}}$ and F and from the results of Section 4.4. The second statement can be proved as in Propositions 6.1 and 6.3.

Regarding the third statement, the linearization of F with respect to directions in $C^\infty_{\boldsymbol{\mu}-1}(\Lambda^1)$ can be computed as in Proposition 6.1. Now choose $y \in T_e\tilde{\mathcal{E}}$ corresponding to a curve $\tilde{e}_s \in \tilde{\mathcal{E}}$ such that $\tilde{e}_0 = e$. Up to identifying $\mathcal{U}$ with M via Φ_L^e, $\Phi_L^{\tilde{e}_s}$ defines a 1-parameter curve of symplectomorphisms ϕ_s of M such that $d/ds(\phi_s)_{|s=0} = v$, for some vector field v on M. Thus, as in Proposition 6.1,

$$\begin{aligned} dF[0](y)\, vol_g &= d/ds(F(\tilde{e}_s, 0)\, vol_g)_{|s=0} = d/ds((\phi_s)^* \operatorname{Im} \Omega)_{|L; s=0} \\ &= (\mathcal{L}_v \operatorname{Im} \Omega)_L = (di_v \operatorname{Im} \Omega)_{|L} \\ &= -d \star \alpha, \end{aligned}$$

where $\alpha := \omega(v,\cdot)_{|L}$ is a closed 1-form on L. Notice that, by definition, $\phi_s \equiv Id$ away from the singularities of L, so $\alpha \equiv 0$ there. Thus, by the Poincaré Lemma (cf. *e.g.* Lemma 2.12), α must be exact on L, *i.e.* $\alpha = d\chi$ for some function $\chi : L \to \mathbb{R}$. We can define χ uniquely by imposing that $\chi \equiv 0$ away from the singularities of L. The function χ depends linearly on y, and we can write $dF[0](y,0) = \Delta_g\,\chi(y)$, as claimed. Furthermore, if $\chi(y) = 0$ then $\alpha = 0$ and $v = 0$. Since $\tilde{\mathcal{E}}$ is defined so as to parametrize geometrically distinct immersions, this implies $y = 0$.

Roughly speaking, near each singularity and up to the appropriate identifications, $\tilde{e}_s$ should be thought of as a 1-parameter curve in the group $\mathrm{SU(m)} \ltimes \mathbb{C}^m$ acting on $\mathbb{C}^m$. This action admits a *moment map* $\mu : \mathbb{C}^m \to (Lie(\mathrm{SU(m)} \ltimes \mathbb{C}^m))^*$. Recall that this means that μ is equivariant and that, for all $w \in Lie(\mathrm{SU(m)} \ltimes \mathbb{C}^m)$, the corresponding function $\mu_w : \mathbb{C}^m \to \mathbb{R}$ satisfies $d\mu_w = i_w\tilde{\omega}$, *i.e.* w is a Hamiltonian vector field with Hamiltonian function μ_w. The moment map can be written explicitly, cf. *e.g.* [6] Section 2.6, showing that each μ_w is at most a quadratic polynomial on $\mathbb{C}^m$. Notice, for future reference, that for any SL $L \subset \mathbb{C}^m$ the calculations in the proof of Proposition 6.1 show that

$$\Delta_g(\mu_{w|L}) = d^*(d\mu_{w|L}) = -\star d \star (i_w\tilde{\omega}_{|L}) = \star(di_w \operatorname{Im}\tilde{\Omega})_{|L} = \star(\mathcal{L}_w \operatorname{Im}\tilde{\Omega})_{|L} = 0,$$

i.e. each μ_w restricts to a harmonic function on L.

In this set-up our vector field v is (locally) an element of $Lie(\mathrm{SU(m)} \ltimes \mathbb{C}^m)$ and $\chi(y) = \mu_v$. Thus $\chi(y)$ is bounded as $r \to 0$. This implies that $\chi(y) \in C^0_{\mathbf{0}}(L)$. Further calculations show that $\chi(y) \in C^\infty_{\mathbf{0}}(L)$, as claimed. □

Now let L be a CS/AC SL in $\mathbb{C}^m$. Define $\tilde{P}$, $\tilde{\mathcal{E}}$, *etc.* analogously to the above (cf. Section 4.4 for the necessary modifications for the ambient space $\mathbb{C}^m$). Let $\mathcal{D}_L$ denote the space of closed 1-forms in $C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)$ whose graph lies in $\mathcal{U}$. Define F as in Equation 6.11.

Proposition 6.5. *Let L be a CS/AC SL in $\mathbb{C}^m$. Then the map F has the following properties:*

1. *The set $F^{-1}(0)$ parametrizes the space of all SL deformations of L which are C^1-close to L away from the singularities and are asymptotic to $\mathcal{C}_i$ with rate $(\boldsymbol{\mu},\boldsymbol{\lambda})$ for some choice of $(\tilde{p}_i,\tilde{v}_i)$ near (p_i,v_i).*
2. *F is a (locally) well-defined smooth map between Fréchet spaces. In particular, for each $\alpha \in \mathcal{D}_L$, $F(\alpha) \in C^\infty_{(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L)$.*
3. *There exists an injective linear map $\chi : T_e\tilde{\mathcal{E}} \to C^\infty_{\mathbf{0}}(L)$ such that (i) $\chi(y) \equiv 0$ away from the singularities and (ii) the linearized map $dF[0] : T_e\tilde{\mathcal{E}} \oplus C^\infty_{(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1) \to C^\infty_{(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L)$ satisfies*

$$dF[0](y,\alpha) = \Delta_g\,\chi(y) + d^*\alpha. \tag{6.13}$$

Proof. The proof is similar to that of Proposition 6.4, but notice that in this case we obtain stronger control over the properties of the function $\chi(y) = \mu_v$, near the singularities: the fact that μ_v restricts to a harmonic function on L implies that $\Delta_g\,\chi(y)$ vanishes in a neighbourhood of each singularity. □

If the spaces $C^\infty(L)$, $C^\infty_{\boldsymbol{\beta}}(L)$ were Banach spaces and the relevant maps were Fredholm, we could now apply the Implicit Function Theorem to conclude that the sets $F^{-1}(0)$, and thus $\mathcal{M}_L$, are smooth. As however they are actually only Fréchet spaces, it is instead necessary to first take the Sobolev space completions of these spaces, then study the Fredholm properties of the linearized maps. We do this in Section 8. This will require some results concerning the Laplace operator on conifolds, summarized in Section 7.

7. Review of the Laplace operator on conifolds

We summarize here some analytic results concerning the Laplace operator on conifolds, referring to [19] for further details and references.

Definition 7.1. Let (Σ, g') be a compact Riemannian manifold. Consider the cone $C := \Sigma \times (0, \infty)$ endowed with the conical metric $\tilde{g} := dr^2 + r^2 g'$. Let $\Delta_{\tilde{g}}$ denote the corresponding Laplace operator acting on functions.

For each component (Σ_j, g'_j) of (Σ, g') and each $\gamma \in \mathbb{R}$, consider the space of homogeneous harmonic functions

$$V^j_\gamma := \{r^\gamma \sigma(\theta) : \Delta_{\tilde{g}}(r^\gamma \sigma) = 0\}. \tag{7.1}$$

Set $m^j(\gamma) := \dim(V^j_\gamma)$. One can show that $m^j_\gamma > 0$ iff γ satisfies the equation

$$\gamma = \frac{(2-m) \pm \sqrt{(2-m)^2 + 4e^j_n}}{2}, \tag{7.2}$$

for some eigenvalue e^j_n of $\Delta_{g'_j}$ on Σ_j. Given any weight $\boldsymbol{\gamma} \in \mathbb{R}^e$, we now set $m(\boldsymbol{\gamma}) := \sum_{j=1}^e m^j(\gamma_j)$. Let $\mathcal{D} \subseteq \mathbb{R}^e$ denote the set of weights $\boldsymbol{\gamma}$ for which $m(\boldsymbol{\gamma}) > 0$. We call these the *exceptional weights* of $\Delta_{\tilde{g}}$.

Let (L, g) be a conifold. Assume (L, g) is asymptotic to a cone $(C, \tilde{g})$ in the sense of Definition 2.6. Roughly speaking, the fact that g is asymptotic to $\tilde{g}$ in the sense of Definition 2.2 implies that the Laplace operator Δ_g is "asymptotic" to $\Delta_{\tilde{g}}$. Applying Definition 7.1 to C defines weights $\mathcal{D} \subseteq \mathbb{R}^e$: we call these the *exceptional weights* of Δ_g. This terminology is due to the following result.

Theorem 7.2. *Let (L, g) be a conifold with e ends. Let $\mathcal{D}$ denote the exceptional weights of Δ_g. Then $\mathcal{D}$ is a discrete subset of $\mathbb{R}^e$ and the Laplace operator*

$$\Delta_g : W^p_{k,\boldsymbol{\beta}}(L) \to W^p_{k-2,\boldsymbol{\beta}-2}(L)$$

is Fredholm iff $\boldsymbol{\beta} \notin \mathcal{D}$.

The above theorem, coupled with the "change of index formula", leads to the following conclusion, cf. [19].

Corollary 7.3. *Let (L, g) be a compact Riemannian manifold. Consider the map $\Delta_g : W^p_k(L) \to W^p_{k-2}(L)$. Then*

$$Im(\Delta_g) = \{u \in W^p_{k-2}(L) : \int_L u \, vol_g = 0\}, \quad Ker(\Delta_g) = \mathbb{R}.$$

Let (L, g) be an AC manifold. Consider the map $\Delta_g : W^p_{k,\boldsymbol{\lambda}}(L) \to W^p_{k-2,\boldsymbol{\lambda}-2}(L)$. If $\boldsymbol{\lambda} > 2 - m$ is non-exceptional then this map is surjective. If $\boldsymbol{\lambda} < 0$ then this map is injective, so for $\boldsymbol{\lambda} \in (2 - m, 0)$ it is an isomorphism.

Let (L, g) be a CS manifold with e ends. Consider the map $\Delta_g : W^p_{k,\boldsymbol{\mu}}(L) \to W^p_{k-2,\boldsymbol{\mu}-2}(L)$. If $\boldsymbol{\mu} \in (2 - m, 0)$ then

$$Im(\Delta_g) = \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u \, vol_g = 0\}, \quad Ker(\Delta_g) = \mathbb{R}.$$

If $\boldsymbol{\mu} > 0$ is non-exceptional then this map is injective and

$$dim(Coker(\Delta_g)) = e + \sum_{0 < \boldsymbol{\gamma} < \boldsymbol{\mu}} m(\boldsymbol{\gamma}),$$

where $m(\boldsymbol{\gamma})$ is as in Definition 7.1.

Let (L,g) be a CS/AC manifold with s CS ends and l AC ends. Consider the map

$$\Delta_g : W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L).$$

If $(\boldsymbol{\mu},\boldsymbol{\lambda}) \in (2-m,0)$ then this map is an isomorphism. If $\boldsymbol{\mu} > 0$ and $\boldsymbol{\lambda} < 0$ are non-exceptional then this map is injective and

$$dim(Coker(\Delta_g)) = s + \sum_{0<\boldsymbol{\gamma}<\boldsymbol{\mu}} m(\boldsymbol{\gamma}),$$

where $m(\boldsymbol{\gamma})$ is as in Definition 7.1. Notice in particular that this dimension depends only on the harmonic functions on the CS cones.

8. Moduli spaces of special Lagrangian conifolds

Recall the statement of the Implicit Function Theorem.

Theorem 8.1. *Let $F : E_1 \to E_2$ be a smooth map between Banach spaces such that $F(0) = 0$. Assume $P := dF[0]$ is surjective and $Ker(P)$ admits a closed complement Z, i.e. $E_1 = Ker(P) \oplus Z$. Then there exists a smooth map $\Phi : Ker(P) \to Z$ such that $F^{-1}(0)$ coincides locally with the graph $\Gamma(\Phi)$ of Φ. In particular, $F^{-1}(0)$ is (locally) a smooth Banach submanifold of E_1.*

The following result is straight-forward.

Proposition 8.2. *Let $F : E_1 \to E_2$ be a smooth map between Banach spaces such that $F(0) = 0$. Assume $P := dF[0]$ is Fredholm. Set $\mathcal{I} := Ker(P)$ and choose Z such that $E_1 = \mathcal{I} \oplus Z$. Let $\mathcal{O}$ denote a finite-dimensional subspace of E_2 such that $E_2 = \mathcal{O} \oplus Im(P)$. Define*

$$G : \mathcal{O} \oplus E_1 \to E_2, \quad (\gamma, e) \mapsto \gamma + F(e).$$

Identify E_1 with $(0, E_1) \subset \mathcal{O} \oplus E_1$. Then:

1. *The map $dG[0] = Id \oplus P$ is surjective and $Ker(dG[0]) = Ker(P)$. Thus, by the Implicit Function Theorem, there exist $\Phi : \mathcal{I} \to \mathcal{O} \oplus Z$ such that $G^{-1}(0) = \Gamma(\Phi)$.*
2. *$F^{-1}(0) = \{(i, \Phi(i)) : \Phi(i) \in Z\} = \{(i, \Phi(i)) : \pi_{\mathcal{O}} \circ \Phi(i) = 0\}$, where $\pi_{\mathcal{O}} : \mathcal{O} \oplus Z \to \mathcal{O}$ is the standard projection.*
3. *Let $\pi_{\mathcal{I}} : \mathcal{I} \oplus Z \to \mathcal{I}$ denote the standard projection. Then $\pi_{\mathcal{I}}$ is a continuous open map so it restricts to a homeomorphism*

$$\pi_{\mathcal{I}} : F^{-1}(0) \to (\pi_{\mathcal{O}} \circ \Phi)^{-1}(0)$$

between $F^{-1}(0)$ and the zero set of the smooth map $\pi_{\mathcal{O}} \circ \Phi : \mathcal{I} \to \mathcal{O}$, which is defined between finite-dimensional spaces.

We now have all the ingredients necessary to prove various smoothness results for SL moduli spaces. In all cases we follow the same steps. Section 6 described each moduli space as the zero set of a map F. The first step is to use regularity to show that one can equivalently study the zero set of a map $\tilde{F}$. The domain of $\tilde{F}$ is of the form $K \times W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L)$ where K is a finite-dimensional vector space defined in terms of spaces introduced in Sections 2.2 and 6. Roughly speaking, this corresponds to separating the obvious Hamiltonian deformations of L from a finite-dimensional space of other Lagrangian deformations. The geometric description of the latter depends on the case in question. The differential $d\tilde{F}[0]$ is then a finite-dimensional perturbation of the Laplace operator Δ_g acting on functions. The second step is to analyze this linearized operator, showing that under appropriate conditions it is surjective. The third step is to identify the kernel of $d\tilde{F}[0]$, at least up to projections. One can then apply the Implicit Function Theorem and conclude.

Smooth compact special Lagrangians. The following result was first proved by McLean [16].

Theorem 8.3. *Let L be a smooth compact SL submanifold of a CY manifold M. Let $\mathcal{M}_L$ denote the moduli space of SL deformations of L. Then $\mathcal{M}_L$ is a smooth manifold of dimension $b^1(L)$.*

Proof. Choose $k \geq 3$ and $p > m$. Consider the space $\mathrm{Ker}(d)$ of closed 1-forms in $W^p_{k-1}(\Lambda^1)$. Let $\mathcal{D}_L$ denote the forms $\alpha \in \mathrm{Ker}(d)$ whose graph $\Gamma(\alpha)$ lies in $\mathcal{U}$. Notice that $\Gamma(\alpha)$ is a well-defined C^1 Lagrangian submanifold in $\mathcal{U}$ by the standard Sobolev embedding $W^p_{k-1}(\Lambda^1) \hookrightarrow C^1(\Lambda^1)$. For the same reason, $\mathcal{D}_L$ is an open neighbourhood of the origin in $\mathrm{Ker}(d)$. Consider the map

$$F : \mathcal{D}_L \to \{u \in W^p_{k-2}(L) : \int_L u\, vol_g = 0\}, \quad \alpha \mapsto \star(\pi_*((\Phi_L^* \operatorname{Im}\Omega)_{|\Gamma(\alpha)})). \tag{8.1}$$

Recall that $W^p_{k-2}(L)$ is closed under multiplication. Together with the ideas of Proposition 6.1, this shows that F is a (locally well-defined) smooth map between Banach spaces with differential $dF[0](\alpha) = d^*\alpha$. Assume $\alpha \in F^{-1}(0)$. Then, by composition with Φ_L, α defines a C^1 SL submanifold in M. Standard regularity results for minimal submanifolds then show that α is smooth. Thus $\mathcal{M}_L$ is locally homeomorphic, via Φ_L, to $F^{-1}(0)$.

Decomposition 1 shows that any $\alpha \in F^{-1}(0)$ is of the form $\alpha = \beta + df$ for some unique $\beta \in H$ and some $f \in C^\infty(L)$, defined up to a constant. We can thus re-phrase the SL deformation problem as follows. Define $\widetilde{\mathcal{D}}_L$ as the space of pairs (β, f) in $H \times W^p_k(L)$ such that $\alpha := \beta + df \in \mathcal{D}_L$. Clearly $\widetilde{\mathcal{D}}_L$ is an open neighbourhood of the origin. Then $\widetilde{\mathcal{D}}_L$ is the domain of the (locally defined) map between Banach spaces

$$\tilde{F} : H \times W^p_k(L) \to \{u \in W^p_{k-2}(L) : \int_L u\, vol_g = 0\}, \quad \tilde{F}(\beta, f) := F(\beta + df). \tag{8.2}$$

Clearly, $d\tilde{F}[0](\beta, f) = d^*\beta + \Delta_g f$. Let $\mathbb{R}$ denote the space of constant functions in $W^p_k(L)$. Notice that both $\widetilde{\mathcal{D}}_L$ and $\tilde{F}$ are invariant under translations in $\mathbb{R}$. Assume $\tilde{F}(\beta, f) = 0$. With respect to f this is a second-order elliptic equation. Standard regularity results show that f is smooth. This proves that $\mathcal{M}_L$ is locally homeomorphic to the quotient space $\tilde{F}^{-1}(0)/\mathbb{R}$. To conclude, it is sufficient to prove that $\tilde{F}^{-1}(0)$ is smooth. According to Corollary 7.3, the map

$$\Delta_g : W^p_k(L) \to \{u \in W^p_{k-2}(L) : \int_L u\, vol_g = 0\} \tag{8.3}$$

is surjective. This implies that $d\tilde{F}[0]$ is surjective. Let β_i be a basis for H. For each β_i the equation $d\tilde{F}[0](\beta_i, f) = 0$ admits a solution f_i. Another solution is given by the pair $\beta = 0$, $f = 1$. It is simple to check that these give a basis for the kernel of $d\tilde{F}[0]$. Applying the Implicit Function Theorem we conclude that $\tilde{F}^{-1}(0)$ is smooth of dimension $b^1(L) + 1$, thus $\mathcal{M}_L$ is smooth of dimension $b^1(L)$. □

AC special Lagrangians. The analogous result for AC SLs was originally proved independently by the author [18] and by Marshall [15]. We present here a simplified proof, starting with the following weighted regularity result due to Joyce, cf. [10] Theorems 5.1 and 7.7.

Lemma 8.4. *Let $\mathcal{C}$ be a SL cone in $\mathbb{C}^m$, endowed with the induced metric $\tilde{g}$ and orientation. Define $\Phi_\mathcal{C} : \mathcal{U} \to \mathbb{C}^m$ and the map F as in Section 6.2. Fix any $\mu > 2$ and $\lambda < 2$ with $\lambda \neq 0$. Assume given a closed 1-form $\alpha \in C^1_{(\mu-1,\lambda-1)}(\mathcal{U})$ satisfying $F(\alpha) = 0$. Analogously to Decomposition 4, we can write $\alpha = \alpha' + dA$ where (i) α' is compactly-supported on the small end and translation-invariant on the large end, and (ii) $A \in C^1_{(\mu,\lambda)}(L)$. Then α' is smooth and $A \in C^\infty_{(\mu,\lambda)}(L)$, so $\alpha \in C^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$.*

Proof. Standard regularity results for minimal submanifolds show that $\alpha \in C^1_{(\mu-1,\lambda-1)}(\mathcal{U}) \cap C^\infty(\mathcal{U})$. Using the same ideas as in the proof of Decomposition 4, this suffices to prove that α' and A are smooth. It is thus enough to show that the higher derivatives of A converge at the correct rate as $r \to \infty$ and $r \to 0$. We sketch here a proof for $r \to \infty$, referring to [10] for details; the other case is analogous.

In terms of A, *i.e.* absorbing the α'-terms into the operator, the equation $F(\alpha) = 0$ corresponds to an equation $\tilde{F}(A) = 0$. Given $r_0 > 0$ and $\epsilon << 1$, consider the equivalent equation

$$r^2 \tilde{F}(A) = 0 \text{ restricted to } \Sigma \times (r_0 - \epsilon r_0, r_0 + \epsilon r_0). \tag{8.4}$$

As in Theorem 8.3 the linearization of $\tilde{F}$ is $\Delta_{\tilde{g}}$. One can check that the change of coordinates $r = e^z$ transforms Equation 8.4 into an equation of the form

$$\Delta_{\tilde{h}}(A) + \cdots = 0 \text{ restricted to } \Sigma \times (r_0' - \epsilon', r_0' + \epsilon'), \tag{8.5}$$

where $\tilde{h}$ is the "cylindrical metric" $\tilde{h} := r^{-2}\tilde{g} = dz^2 + g'$. Up to a translation we can identify $\Sigma \times (r_0' - \epsilon', r_0' + \epsilon')$ with the fixed, *i.e.* r_0-independent, domain $\Sigma \times (-\epsilon', \epsilon')$. One can show that Equation 8.5 converges to the equation $\Delta_{\tilde{h}}(A) = 0$ on this domain in such a way that interior estimates for the solutions are uniform as $r_0 \to \infty$. In particular, in terms of Hölder norms, there exists a constant $C = C(k, \beta)$ independent of r_0 such that

$$\|A\|_{C^{k,\beta}} \leq C \cdot \|A\|_{C^1} \tag{8.6}$$

on the domain $\Sigma \times (-\epsilon', \epsilon')$ and with respect to the metric $\tilde{h}$. To be precise, as this is an "interior" estimate, the domain on the left hand side is slightly smaller than the domain on the right hand side.

Let us now write this estimate in terms of the coordinate r and multiply both sides by $r^{-\lambda}$. We can then check that

$$\|A\|_{C^k_\lambda} \leq C \cdot \|A\|_{C^1_\lambda} \tag{8.7}$$

on the domain $\Sigma \times (r_0 - \epsilon r_0, r_0 + \epsilon r_0)$ and with respect to the metric $\tilde{g}$. As r_0 is arbitrary and $\|A\|_{C^1_\lambda}$ is bounded on the large end, this shows that $\|A\|_{C^k_\lambda}$ is bounded for all k so $A \in C^\infty_\lambda$. ☐

Theorem 8.5. *Let L be an AC SL submanifold of $\mathbb{C}^m$ with rate $\boldsymbol{\lambda}$. Let $\mathcal{M}_L$ denote the moduli space of SL deformations of L with rate $\boldsymbol{\lambda}$. Consider the operator*

$$\Delta_g : W^p_{k,\boldsymbol{\lambda}}(L) \to W^p_{k-2,\boldsymbol{\lambda}-2}(L). \tag{8.8}$$

1. *If $\boldsymbol{\lambda} \in (0,2)$ is a non-exceptional weight for Δ_g then $\mathcal{M}_L$ is a smooth manifold of dimension $b^1(L) + dim(Ker(\Delta_g)) - 1$.*
2. *If $\boldsymbol{\lambda} \in (2-m, 0)$ then $\mathcal{M}_L$ is a smooth manifold of dimension $b^1_c(L)$.*

Proof. As in Theorem 8.3, choose $k \geq 3$ and $p > m$ so that $W^p_{k-1,\boldsymbol{\lambda}-1}(\Lambda^1) \subset C^1_{\boldsymbol{\lambda}-1}(\Lambda^1)$. Let $\mathcal{D}_L$ denote the space of closed 1-forms in $W^p_{k-1,\boldsymbol{\lambda}-1}(\Lambda^1)$ whose graph $\Gamma(\alpha)$ lies in $\mathcal{U}$. Consider the map

$$F : \mathcal{D}_L \to W^p_{k-2,\boldsymbol{\lambda}-2}(L), \quad \alpha \mapsto \star(\pi_*((\Phi_L^* \operatorname{Im} \tilde{\Omega})_{|\Gamma(\alpha)})). \tag{8.9}$$

Assume $\boldsymbol{\lambda} < 2$. In this case Theorem 2.10 shows that $W^p_{k-2,\boldsymbol{\lambda}-2}(L)$ is closed under multiplication. Together with the ideas of Proposition 6.1, this shows that F is a (locally well-defined) smooth map between Banach spaces with differential $dF[0](\alpha) = d^*\alpha$. Assume $F(\alpha) = 0$. Theorem 2.10 and Lemma 8.4 then show that $\alpha \in C^\infty_{\boldsymbol{\lambda}-1}(\Lambda^1)$ so $F^{-1}(0)$ is locally homeomorphic, via Φ_L, to $\mathcal{M}_L$.

Now assume $\boldsymbol{\lambda} \in (0,2)$. Decomposition 2 shows that any $\alpha \in F^{-1}(0)$ is of the form $\alpha = \beta + df$, for some $\beta \in H$ and some $f \in C^\infty_{\boldsymbol{\lambda}}(L)$. Define $\widetilde{\mathcal{D}}_L$ as the space of pairs (β, f) in

$H \times W^p_{k,\boldsymbol{\lambda}}(L)$ such that $\alpha := \beta + df \in \mathcal{D}_L$. Clearly $\widetilde{\mathcal{D}}_L$ is an open neighbourhood of the origin. Then $\widetilde{\mathcal{D}}_L$ is the domain of the (locally defined) smooth map between Banach spaces

$$\tilde{F} : H \times W^p_{k,\boldsymbol{\lambda}}(L) \to W^p_{k-2,\boldsymbol{\lambda}-2}(L), \quad \tilde{F}(\beta, f) := F(\beta + df) \tag{8.10}$$

with $d\tilde{F}[0](\beta, f) = d^*\beta + \Delta_g f$ and invariant under translations in $\mathbb{R}$. Assume $\tilde{F}(\beta, f) = 0$. Theorem 2.10 and Lemma 8.4 then show that $f \in C^\infty_{\boldsymbol{\lambda}}(L)$. This proves that $\mathcal{M}_L$ is locally homeomorphic, via Φ_L, to the quotient space $\tilde{F}^{-1}(0)/\mathbb{R}$. To conclude, it is thus sufficient to prove that $\tilde{F}^{-1}(0)$ is smooth. For this we need to further assume that $\boldsymbol{\lambda}$ is non-exceptional. Then Corollary 7.3 shows that the map of Equation 8.8 is surjective, so $d\tilde{F}[0]$ is surjective. Let β_i be a basis for H. For each β_i the equation $d\tilde{F}[0](\beta_i, f) = 0$ admits a solution f_i. More solutions are given by the pairs $\beta = 0$, $f \in Ker(\Delta_g)$. It is simple to check that these give a basis for the kernel of $d\tilde{F}[0]$. Applying the Implicit Function Theorem we conclude that $\tilde{F}^{-1}(0)$ is smooth of dimension $\dim(H \oplus \mathrm{Ker}(\Delta_g))$. Thus $\mathcal{M}_L$ is smooth and has the claimed dimension.

Now assume $\boldsymbol{\lambda} \in (2-m, 0)$. In this case Decomposition 3 shows that any $\alpha \in F^{-1}(0)$ is of the form $\alpha = \beta + dv + df$, for some $\beta \in \widetilde{H}_\infty$, $dv \in d(E_\infty)$ and $df \in d(C^\infty_{\boldsymbol{\lambda}}(L))$. We can use regularity as before to prove that $\mathcal{M}_L$ is locally homeomorphic to the quotient space $\tilde{F}^{-1}(0)/\mathbb{R}$, for the (locally defined) map

$$\tilde{F} : \widetilde{H}_\infty \times E_\infty \times W^p_{k,\boldsymbol{\lambda}}(L) \to W^p_{k-2,\boldsymbol{\lambda}-2}(L), \quad \tilde{F}(\beta, v, f) = F(\beta + dv + df). \tag{8.11}$$

Notice that this time the constant functions $\mathbb{R}$ are contained in E_∞. We conclude as before that $\tilde{F}^{-1}(0)$ is smooth, this time of dimension $\dim(\widetilde{H}_\infty \oplus E_\infty)$. Remark 2.13 then shows that $\mathcal{M}_L$ is smooth of dimension $b^1_c(L)$. □

CS special Lagrangians. Now assume that L is CS SL with singularities modelled on cones $\mathcal{C}_i$. It turns out that smoothness of $\mathcal{M}_L$ then requires an additional "stability" assumption on $\mathcal{C}_i$. Roughly speaking, it is required that the cones $\mathcal{C}_i$ admit no additional harmonic functions with prescribed growth, beyond those which necessarily exist for geometric reasons.

Definition 8.6. Let $\mathcal{C}$ be a SL cone in $\mathbb{C}^m$. Let (Σ, g') denote the link of $\mathcal{C}$ with the induced metric. Assume $\mathcal{C}$ has a unique singularity at the origin; equivalently, assume that Σ is smooth and that it is not a sphere $\mathbb{S}^{m-1} \subset \mathbb{S}^{2m-1}$. Recall from the proof of Proposition 6.4 that the standard action of $\mathrm{SU(m)} \ltimes \mathbb{C}^m$ on $\mathbb{C}^m$ admits a moment map μ and that the components of μ restrict to harmonic functions on $\mathcal{C}$. Let G denote the subgroup of SU(m) which preserves $\mathcal{C}$. Then μ defines on $\mathcal{C}$ $2m$ linearly independent harmonic functions of linear growth; in the notation of Definition 7.1 these functions are contained in the space V_γ with $\gamma = 1$. The moment map also defines on $\mathcal{C}$ $m^2 - 1 - \dim(G)$ linearly independent harmonic functions of quadratic growth: these belong to the space V_γ with $\gamma = 2$. Constant functions define a third space of homogeneous harmonic functions on $\mathcal{C}$, *i.e.* elements in V_γ with $\gamma = 0$. In particular, these three values of γ are always exceptional values for the operator $\Delta_{\tilde{g}}$ on any SL cone, in the sense of Definition 7.1.

We say that $\mathcal{C}$ is *stable* if these are the only functions in V_γ for $\gamma = 0, 1, 2$ and if there are no other exceptional values γ in the interval $[0, 2]$. More generally, let L be a CS or CS/AC SL submanifold. We say that a singularity x_i of L is *stable* if the corresponding cone $\mathcal{C}_i$ is stable.

The following result is due to Joyce [11].

Theorem 8.7. *Let L be a CS SL submanifold of M with s singularities and rate $\boldsymbol{\mu}$. Let $\mathcal{M}_L$ denote the moduli space of SL deformations of L with moving singularities and rate $\boldsymbol{\mu}$. Assume*

$\boldsymbol{\mu}$ is non-exceptional for the map

$$\Delta_g : W^p_{k,\boldsymbol{\mu}}(L) \to \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u\, vol_g = 0\}. \tag{8.12}$$

Then $\mathcal{M}_L$ is locally homeomorphic to the zero set of a smooth map $\Phi : \mathcal{I} \to \mathcal{O}$ defined (locally) between finite-dimensional vector spaces. If $\boldsymbol{\mu} = 2 + \epsilon$ and all singularities are stable then $\mathcal{O} = \{0\}$ and $\mathcal{M}_L$ is smooth of dimension $dim(\mathcal{I}) = b^1_c(L) - s + 1$.

Proof. Start with a map F defined as in Section 6.3 on $\tilde{\mathcal{E}} \times W^p_{k-1,\boldsymbol{\mu}-1}(\Lambda^1)$. As in Theorem 8.5, regularity and Decomposition 3 show that $\mathcal{M}_L$ is locally homeomorphic to $\tilde{F}^{-1}(0)/\mathbb{R}$, where $\tilde{F}$ is the (locally-defined) map

$$\begin{aligned} \tilde{F} : \tilde{\mathcal{E}} \times \widetilde{H}_0 \times E_0 \times W^p_{k,\boldsymbol{\mu}}(L) &\to \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u\, vol_g = 0\} \\ (\tilde{e}, \beta, v, f) &\mapsto F(\tilde{e}, \beta + dv + df), \end{aligned}$$

invariant under translations in $\mathbb{R} \subset E_0$. As in Proposition 6.4, $d\tilde{F}[0](y, \beta, v, f) = d^*\beta + \Delta_g(\chi(y) + v + f)$. Now consider the restricted map

$$d\tilde{F}[0] : T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,\boldsymbol{\mu}}(L) \to \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u\, vol_g = 0\}. \tag{8.13}$$

We claim that the kernel of this map is given by the constant functions $\mathbb{R}$. To prove this, assume $d\tilde{F}[0](\chi(y) + v + f) = 0$. Since $\chi(y) + v + f \in W^p_{k,-\boldsymbol{\epsilon}}(L)$, Corollary 7.3 shows that $\chi(y) + v + f$ is constant, *i.e.* $d(\chi(y) + v + f) = 0$. In other words the infinitesimal Lagrangian deformation of L defined by (y, v, f) is trivial, so in particular $y = 0$. This implies $\chi(y) = 0$ and it is simple to conclude that $f = 0$ and $v \in \mathbb{R}$.

Let $\mathcal{O}$ denote the cokernel of the map of Equation 8.13. More precisely, we define it to be a finite-dimensional space of $W^p_{k-2,\mu-2}(L)$ such that

$$\mathcal{O} \oplus d\tilde{F}[0]\left(T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,\boldsymbol{\mu}}\right) = \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u\, vol_g = 0\}. \tag{8.14}$$

Consider the map

$$\begin{aligned} G : \mathcal{O} \times \tilde{\mathcal{E}} \times \widetilde{H}_0 \times E_0 \times W^p_{k,\boldsymbol{\mu}}(L) &\to \{u \in W^p_{k-2,\boldsymbol{\mu}-2}(L) : \int_L u\, vol_g = 0\} \\ (\gamma, \tilde{e}, \beta, v, f) &\mapsto \gamma + \tilde{F}(\tilde{e}, \beta, v, f). \end{aligned}$$

Again, G is invariant under translations in $\mathbb{R}$. By construction, the restriction of $dG[0]$ to the space $\mathcal{O} \oplus T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,\boldsymbol{\mu}}$ is surjective with kernel $\mathbb{R}$. We now have the following information about the map G. Firstly, $\mathrm{Ker}(dG[0]) = V \oplus \mathbb{R}$, where V is some vector space projecting isomorphically onto $\widetilde{H}_0$. Secondly, by the Implicit Function Theorem, the set $G^{-1}(0)$ is smooth and can be locally written as the graph of a smooth map Φ defined on the kernel of $dG[0]$, thus on $\widetilde{H}_0 \oplus \mathbb{R}$. As in Proposition 8.2 we can conclude that the projection onto $\widetilde{H}_0 \oplus \mathbb{R}$ restricts to a homeomorphism $\tilde{F}^{-1}(0) \simeq (\pi_{\mathcal{O}} \circ \Phi)^{-1}(0)$. It is simple to check that Φ is invariant under translations in $\mathbb{R}$. Restricting Φ to $\mathcal{I} := \widetilde{H}_0$ proves the first claim.

Now let us further assume that $\boldsymbol{\mu} = 2 + \epsilon$ and that all singularities are stable. Here, ϵ is to be understood as in Remark 5.5; in particular, the moduli space we will obtain is independent of the particular ϵ chosen. Recall from Corollary 7.3 that for $\boldsymbol{\mu} > 2 - m$ we can compute the dimension of $\mathrm{Coker}(\Delta_g)$ in terms of the number of harmonic functions on the cones $\mathcal{C}_i$. Recall

from Definition 8.6 that SL cones always admit a certain number of harmonic functions. This implies that, for the operator $\Delta_g : W^p_{k,\boldsymbol{\mu}}(L) \to W^p_{k-2,\boldsymbol{\mu}-2}(L)$,

$$\dim(\mathrm{Coker}(\Delta_g)) \geq d, \quad \text{where } d := \sum_{i=1}^{e} \left(1 + 2m + m^2 - 1 - \dim(G_i)\right). \tag{8.15}$$

The stability condition is equivalent to $\dim(\mathrm{Coker}(\Delta_g)) = d$. This means that the cokernel of the operator in Equation 8.12 has dimension $d-1$. Notice that d is also the dimension of the space $T_e\tilde{\mathcal{E}} \oplus E_0$. Our calculation of the kernel thus implies that the map $d\tilde{F}[0]$ of Equation 8.13 is surjective. Thus $\mathcal{O} = \{0\}$. We can now apply the Implicit Function Theorem directly to $\tilde{F}$ to obtain that $\tilde{F}^{-1}(0)$ is smooth, of dimension $\dim(\widetilde{H}_0) + 1$. Quotienting by $\mathbb{R}$ and using Equation 2.6 gives the desired result. □

We call $\mathcal{O}$ the *obstruction space* of the SL deformation problem.

CS/AC special Lagrangians in $\mathbb{C}^m$. We can now state and prove the main result of this paper.

Theorem 8.8. *Let L be a CS/AC SL submanifold of $\mathbb{C}^m$ with s CS ends, l AC ends and rate $(\boldsymbol{\mu}, \boldsymbol{\lambda})$. Let $\mathcal{M}_L$ denote the moduli space of SL deformations of L with moving singularities and rate $(\boldsymbol{\mu}, \boldsymbol{\lambda})$. Assume $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ is non-exceptional for the map*

$$\Delta_g : W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L). \tag{8.16}$$

We will restrict our attention to the two cases $\boldsymbol{\lambda} \in (2-m, 0)$ or $\boldsymbol{\lambda} \in (0, 2)$. In either case $\mathcal{M}_L$ is locally homeomorphic to the zero set of a smooth map $\Phi : \mathcal{I} \to \mathcal{O}$ defined (locally) between finite-dimensional vector spaces. If furthermore $\boldsymbol{\mu} = 2 + \epsilon$ and all singularities are stable then $\mathcal{O} = \{0\}$ and $\mathcal{M}_L$ is smooth of dimension $\dim(\mathcal{I})$. Specifically:

1. *If $\boldsymbol{\lambda} \in (2-m, 0)$ then $\dim(\mathcal{I}) = b^1_c(L) - s$.*
2. *If $\boldsymbol{\lambda} \in (0, 2)$ then $\dim(\mathcal{I}) = b^1_{c,\bullet}(L) - s + \sum_{i=1}^{l} d_i$, where d_i is the number of harmonic functions on the AC end S_i of the form $r^\gamma \sigma(\theta)$ with $\gamma \in [0, \lambda_i]$.*

Proof. Start with a map F defined as in the previous theorems on $\tilde{\mathcal{E}} \times W^p_{k-1,(\boldsymbol{\mu}-1,\boldsymbol{\lambda}-1)}(\Lambda^1)$, such that $\mathcal{M}_L \simeq F^{-1}(0)$. Let $\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ denote the map of Equation 8.16.

We split the proof into two parts, depending on the range of $\boldsymbol{\lambda}$. To begin, assume $\boldsymbol{\lambda} \in (2-m, 0)$. By regularity and Decomposition 4, $\mathcal{M}_L$ is locally homeomorphic to $\tilde{F}^{-1}(0)/\mathbb{R}$, where $\tilde{F}$ is the (locally-defined) map

$$\begin{array}{rcl} \tilde{F} : \tilde{\mathcal{E}} \times \widetilde{H}_{0,\infty} \times E_{0,\infty} \times W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) & \to & W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L) \\ (\tilde{e}, \beta, v, f) & \mapsto & F(\tilde{e}, \beta + dv + df). \end{array}$$

As in Proposition 6.5, $d\tilde{F}[0](y, \beta, v, f) = d^*\beta + \Delta_g(\chi(y) + v + f)$. Now consider the restricted map

$$d\tilde{F}[0] : T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L), \tag{8.17}$$

where E_0 is the subspace of functions in $E_{0,\infty}$ which vanish on the AC ends. Notice that $\chi(y) + v + f \in W^p_{k,(-\boldsymbol{\epsilon},\boldsymbol{\lambda})}(L)$. As in Theorem 8.7 we can use Corollary 7.3 to prove that the map of Equation 8.17 is injective.

Let $\mathcal{O}$ denote the cokernel of the map of Equation 8.17. More precisely, we define it to be a finite-dimensional subspace of $W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L)$ such that

$$\mathcal{O} \oplus d\tilde{F}[0]\left(T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}\right) = W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L). \tag{8.18}$$

Consider the map

$$\begin{aligned} G : \mathcal{O} \times \tilde{\mathcal{E}} \times \widetilde{H}_{0,\infty} \times E_{0,\infty} \times W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) &\to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L) \\ (\gamma, \tilde{e}, \beta, v, f) &\mapsto \gamma + \tilde{F}(\tilde{e}, \beta, v, f). \end{aligned}$$

By construction the restriction of $dG[0]$ to the space $\mathcal{O} \oplus T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}$ is an isomorphism. Let E' denote a complement of $E_0 \oplus \mathbb{R}$ in $E_{0,\infty}$, *i.e.* $E_{0,\infty} = E_0 \oplus \mathbb{R} \oplus E'$. As in Theorem 8.7, $G^{-1}(0)$ is smooth and can be written as the graph of a smooth map Φ defined on $\widetilde{H}_{0,\infty} \oplus (\mathbb{R} \oplus E')$. Restricting Φ to $\mathcal{I} := \widetilde{H}_{0,\infty} \oplus E'$ and using the same arguments as in Proposition 8.2 and Theorem 8.7 then proves the first claim regarding $\mathcal{M}_L$ for this range of $\boldsymbol{\lambda}$. Notice that $\dim(\widetilde{H}_{0,\infty}) = b^1_c(L) - (s+l) + 1$ and $\dim(E') = l - 1$ so $\dim(\mathcal{I}) = b^1_c(L) - s$.

Now let us further assume that $\boldsymbol{\mu} = 2 + \epsilon$ and that all singularities are stable. Here, as in Theorem 8.7, ϵ is to be understood as in Remark 5.5. By Corollary 7.3 and the definition of stability,

$$\dim(\operatorname{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})) = d, \quad \text{where } d := \sum_{i=1}^{s} \left(1 + 2m + m^2 - 1 - \dim(G_i)\right). \tag{8.19}$$

Again, d is also the dimension of the space $T_e\tilde{\mathcal{E}} \oplus E_0$. Our previous injectivity calculation thus implies that the map $d\tilde{F}[0]$ of Equation 8.17 is an isomorphism. In particular, $\mathcal{O} = \{0\}$. We can now apply the Implicit Function Theorem directly to $\tilde{F}$ to obtain that $\tilde{F}^{-1}(0)$ is smooth. Quotienting by $\mathbb{R}$ shows that $\mathcal{M}_L$ is smooth.

We now start over again, under the assumption $\boldsymbol{\lambda} \in (0,2)$. In this case we use the map

$$\begin{aligned} \tilde{F} : \tilde{\mathcal{E}} \times \widetilde{H}_{0,\bullet} \times E_0 \times W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) &\to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L) \\ (\tilde{e}, \beta, v, f) &\mapsto F(\tilde{e}, \beta + dv + df) \end{aligned}$$

and the restricted map

$$d\tilde{F}[0] : T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L). \tag{8.20}$$

Recall the construction of E_0 in Decomposition 4: it is clear that we may assume that $\chi(T_e\tilde{\mathcal{E}})$ and E_0 are linearly independent in $W^p_{k,(-\boldsymbol{\epsilon},-\boldsymbol{\epsilon})}(L)$. Corollary 7.3 proves that Δ_g is injective on this space. Define a decomposition

$$T_e\tilde{\mathcal{E}} \oplus E_0 = Z' \oplus Z'' \tag{8.21}$$

by imposing $\Delta_g(Z') = \Delta_g(T_e\tilde{\mathcal{E}} \oplus E_0) \cap \operatorname{Im}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$ and choosing any complement Z''. Then one can check that the kernel of the map of Equation 8.20 is isomorphic to $Z' \oplus \operatorname{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$.

Choose $\mathcal{O}$ in $W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L)$ such that

$$\mathcal{O} \oplus d\tilde{F}[0]\left(T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}\right) = W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L). \tag{8.22}$$

Consider the map

$$\begin{aligned} G : \mathcal{O} \times \tilde{\mathcal{E}} \times \widetilde{H}_{0,\bullet} \times E_0 \times W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}(L) &\to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}-2)}(L) \\ (\gamma, \tilde{e}, \beta, v, f) &\mapsto \gamma + \tilde{F}(\tilde{e}, \beta, v, f). \end{aligned}$$

The restriction of $dG[0]$ to the space $\mathcal{O} \oplus T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda})}$ is surjective. As before, this implies that $G^{-1}(0)$ can be parametrised via a smooth map Φ defined (locally) on the space $\widetilde{H}_{0,\bullet} \oplus Z' \oplus \operatorname{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$. As usual, these maps are invariant under translations in $\mathbb{R} \subset Z' \oplus \operatorname{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$. Setting $\mathcal{I} := (\widetilde{H}_{0,\bullet} \oplus Z' \oplus \operatorname{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}))/\mathbb{R}$ and considering the natural map on this quotient then proves the first claim regarding $\mathcal{M}_L$ for this range of $\boldsymbol{\lambda}$.

Now assume that $\boldsymbol{\mu} = 2 + \epsilon$ and that all singularities are stable. Choose $\boldsymbol{\lambda}' \in (2-m, 0)$. We can restrict the map of Equation 8.20 to the map

$$d\tilde{F}[0] : T_e\tilde{\mathcal{E}} \oplus E_0 \oplus W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda}')}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}'-2)}(L). \tag{8.23}$$

Exactly as for Equation 8.17, it is simple to prove that Equation 8.23 defines an isomorphism and that $\dim(T_e\tilde{\mathcal{E}} \oplus E_0) = \dim(\mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'}))$, where

$$\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'} := \Delta_g : W^p_{k,(\boldsymbol{\mu},\boldsymbol{\lambda}')}(L) \to W^p_{k-2,(\boldsymbol{\mu}-2,\boldsymbol{\lambda}'-2)}(L).$$

One can check that the dimension of $\mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$ decreases as $\boldsymbol{\lambda}$ increases. We can actually assume, cf. [19], that $\mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}) \subseteq \mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'})$. This proves that the map of Equation 8.20 is surjective, *i.e.* $\mathcal{O} = \{0\}$, so $\tilde{F}^{-1}(0)$ and $\mathcal{M}_L$ are smooth. To compute the dimension of this moduli space notice that $Z'' \simeq \mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})$ so

$$\begin{aligned}
\dim(\mathrm{Ker}(d\tilde{F}[0])) &= \dim(\mathrm{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})) + \dim(Z') \\
&= \dim(\mathrm{Ker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})) + \dim(\mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'})) - \dim(\mathrm{Coker}(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}})) \\
&= i(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}) - i(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'}),
\end{aligned} \tag{8.24}$$

where i denotes the index of the Fredholm map. This implies that the kernel of the full map $d\tilde{F}[0]$ has dimension $\dim(\widetilde{H}_{0,\bullet}) + i(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}) - i(\Delta_{\boldsymbol{\mu},\boldsymbol{\lambda}'})$. The conclusion follows from Equation 2.16 and the change of index formula, cf. [19]. □

Remark 8.9. Notice that, when $\boldsymbol{\lambda} < 0$ and the stability condition is verified, the dimension of the SL moduli spaces appearing in Theorems 8.3, 8.5, 8.7 and 8.8 is purely topological. The cases analyzed in the theorems correspond exactly to the cases analyzed in Corollary 2.18, in the sense that the moduli spaces should be thought of as being modelled on the cohomology spaces which appear in Corollary 2.18.

It is interesting to notice how decay conditions on AC and CS ends are incorporated differently into these cohomology spaces: decay conditions on AC ends correspond to using compactly-supported forms while decay conditions on CS ends correspond to the condition that a certain restriction map vanishes, cf. also Remark 2.19.

Allowing $\boldsymbol{\lambda} > 0$ changes the topological data, again in agreement with Corollary 2.18. It also introduces new SL deformations which depend on analytic data.

Example 8.10. Let $\mathcal{C}$ be a SL cone in $\mathbb{C}^m$. Assume $\mathcal{C}$ is stable and that its link Σ is connected so that $s = 1$. Using Poincaré Duality and the fact that $\mathcal{C} \simeq \Sigma \times (0, \infty)$ we see that

$$b^1_c(\mathcal{C}) = b^{m-1}(\mathcal{C}) = b^{m-1}(\Sigma) = 1. \tag{8.25}$$

Theorem 8.8 then shows that, for $\lambda \in (2-m, 0)$, $\mathcal{M}_{\mathcal{C}}$ has dimension 0, *i.e.* $\mathcal{C}$ is rigid within this class of deformations.

Notice also that restriction defines isomorphisms $H^i(\mathcal{C};\mathbb{R}) \simeq H^i(\Sigma;\mathbb{R})$ so the long exact sequence 2.16, using $\Sigma_0 = \Sigma$, leads to $H^i_{c,\bullet}(\mathcal{C};\mathbb{R}) = 0$. Theorem 8.8 then shows that $\mathcal{M}_{\mathcal{C}}$ has dimension 0 if $\lambda \in (0,1)$ and has dimension $2m$ if $\lambda \in (1,2)$. In the latter case the SL deformations are simply the translations of $\mathcal{C}$ in $\mathbb{C}^m$.

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